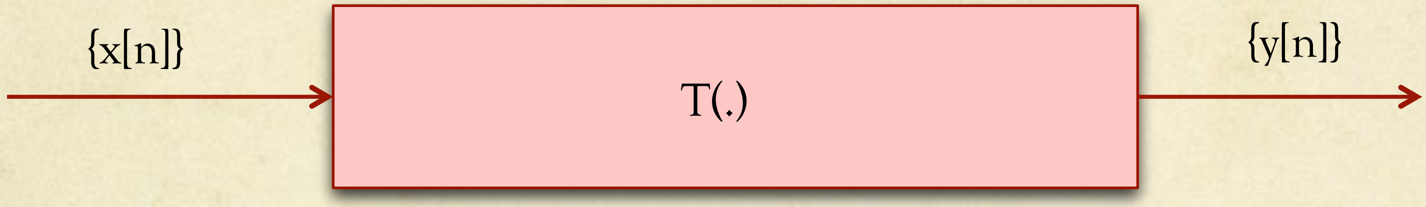
Digital Signal Processing Discrete Time Signal & Systems

Lecture-2

16-March-16

Discrete-Time Systems

- It is a transformation or operator that maps an input sequence {x[n]} to an output sequence {y[n]}.



- Where T(.) define different classes of systems.

Basic System Properties

- **<u>Systems with or without memory</u>**:
  - A system is said to be memory less if the output for each value of the independent variable at a given time n depends only on the input value at time n.
    - E.g., $y[n] = \cos(x[n])+3$ is memory less.
    - A particularly simple memory less system is the identity system defined by: $y[n]=x[n]$.
  - Not all systems are memory less. A simple example of system with memory is a delay defined by: $y[n] = x[n-1]$.
  - A system with memory retains or stores information about input values at times other than the current input value.

Basic System Properties (cont.)

- **Invertibility**:
  - A system is said to be invertible if the input signal {x[n]} can be recovered from the output signal {y[n]}.
  - For this to be true two different input signals should produce two different outputs.
  - If some different input signal produce same output signal then by processing output we can not say which input produced the output.
    - E.g., $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
    - Then, x[n] = y[n] – y[n-1]
  - For example if a non-invertible system is: y[n]=0

Basic System Properties (cont.)

- **<u>Invertibility</u>**: (cont.)
  - The system produces an all zero sequence for any input sequence.
  - Since every input sequence gives all zero sequence.
  - We cannot find out which input produced the output.
  - The system which produces the sequence {x[n]} form sequence {y[n]} is called the inverse system.

Basic System Properties (cont.)

- **Casuality**:
  - A system is casual if the output at anytime depends only on values of the input at the present time and in the past. E.g., y[n]=f(x[n],x[n-1],….)
  - All memory less systems are casual.
  - The system defined by: $y[n]=\frac{1}{2N+1}\sum_{k=-N}^{N} x[n-k]$ is non-casual.

Basic System Properties (cont.)

- **Stability**:
  - A system is said to be BIBO stable if every bounded input produces a bounded output.
  - The signal x[n] is bounded if: $|x[n]| < M < \infty$, for all n.
  - The moving average system $y[n] = \frac{1}{2N+1}\sum_{k=-N}^{N} x[n]$ is stable as y[n] is sum of finite numbers and so it is bounded.
  - The system defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$ is unstable.

Basic System Properties (cont.)

- **Time Invariance**:
  - A system is said to be time invariant if the behavior and characteristics of the system do not change with time.
  - Thus a system is said to be time invariant if a time delay or time advance in the input signal leads to identical delay or advance in the output signal.
  - Mathematically if:

$$\{y[n]\} = T(\{x[n]\})$$

$$\textit{then} \quad \{y[n-n_0]\} = T(\{x[n-n_0]\}) \quad \textit{for} \quad \textit{any} \quad n_0$$

Basic System Properties (cont.)

- **Linearity**:
  - Is a system satisfies the property of homogeneity and the property of superposition, the system is said to be linear.
  - Homogeneity:
    - Let the system produce an output y(t) for an input x(t).
    - If the input is scaled 'a' times, the output will also be scaled 'a' times for all values of 'a' and x(t).
    - Mathematically we can write as follows:

$$\text{if} \quad x(t) \rightarrow y(t)$$

$$ax(t) \rightarrow ay(t)$$

Basic System Properties (cont.)

- **Linearity**:
  - Superposition:
    - If two inputs $x_1(t)$ and $x_2(t)$ are applied to a system simultaneously, the overall output will be the effects of $x_1(t)$ and $x_2(t)$.

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

    - Combining these properties we can write following for the linear system:

$$if \quad x_1(t) \rightarrow y_1(t) \quad and \quad x_2(t) \rightarrow y_2(t), \quad then$$

$$ax_1(t) + ax_2(t) \rightarrow ay_1(t) + ay_2(t)$$

Basic System Properties (cont.)

- **Non-Linear System**:
  - If any system does not satisfy the above two conditions, the system is called as non-linear.

Basic System Properties (cont.)

- **<u>Linear Time Variant & Linear Time Invariant</u>:**
  - If a system is both linear and time variant, then it is known as linear time variant system.
  - If a system is both linear and time invariant then such a system is known as linear time invariant system.

Convolution Sum

- The property of time invariance implies that if h[n] is the response to δ[n], then the response to δ[n-k] is h[n-k]. Hence,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- A linear time invariant system is completely characterized by its impulse response h[n] in the sense that, given h[n], it is possible to use above equation to compute the output y[n] due to any input x[n].

- The above equation is called the convolution sum.

- y[n] is the convolution of x[n] with h[n] and represents this by the notation: y[n] = x[n]*h[n].

- The operation of discrete-time convolution takes two sequences x[n] and h[n] and produces a third sequence y[n].

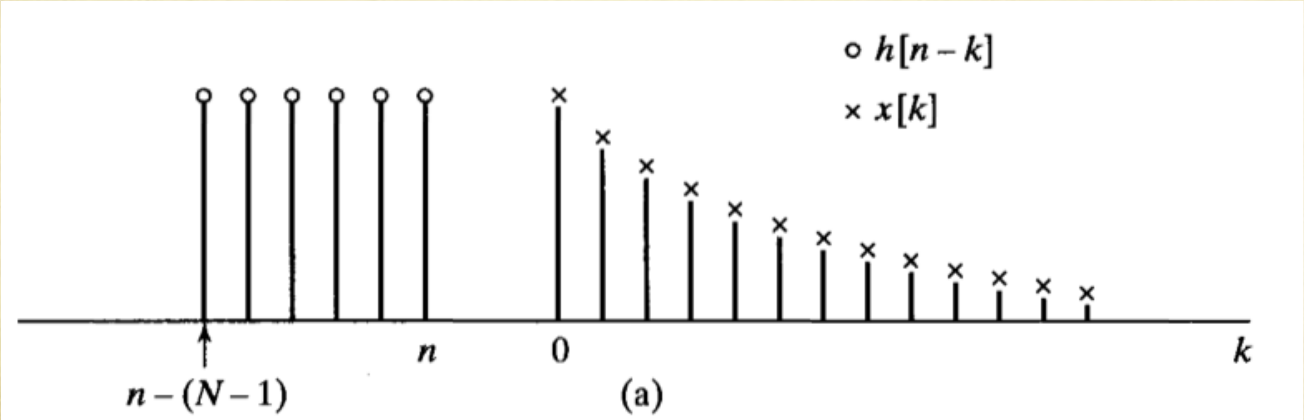
Example #1

- Consider a system with impulse response:

$$h[n] = u[n] - u[n-N]$$

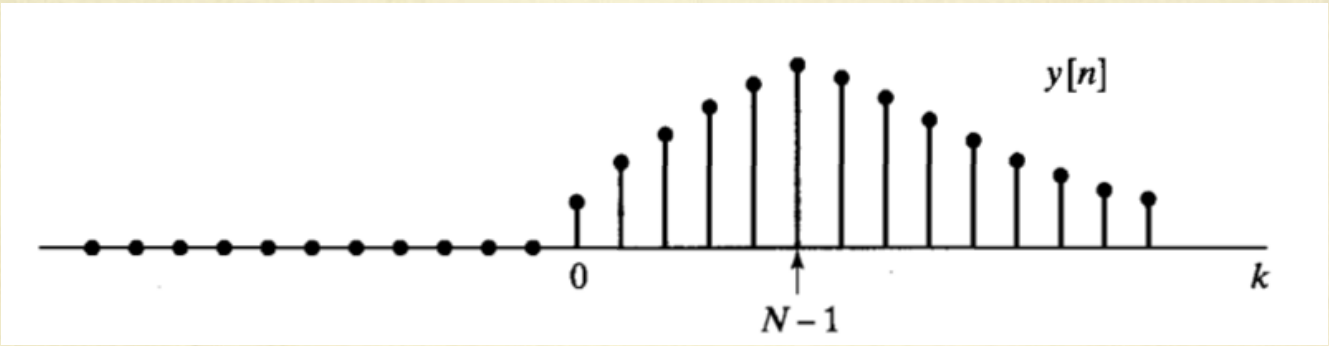
$$= \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

- The input is $x[n]=a^n u[n]$



Example #1 (cont.)

- Solution:



$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1 \\ a^{n-N+1}\left(\dfrac{1-a^{N}}{1-a}\right), & N-1 < n \end{cases}$$

DT Impulse Response of LTI System

- If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses.
- A linear system can be completely characterized by its impulse response.
- Let x[n] be the input signal and y[n] be the output sequence and T() represent the linear system.

$$y[n] = T(x[n])$$

$$= T\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right)$$

DT Impulse Response of LTI System (cont.)

- From the principle of superposition we can write:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] T\left\{\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$$

- The system response to any input can be expressed in terms of the responses of the system to the sequence δ [n-k].
- If only linearity is imposed, $h_k[n]$ will depend on both n and k, in which case the computational usefulness is limited.
- The more useful result is obtained if the additional constraint of time invariance is imposed.

DT Impulse Response of LTI System (cont.)

- The property of time invariance implies that if h[n] is the response to δ[n], then the response to δ[n-k] is h[n-k].
- With this additional constraint the above equation becomes:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- A LTI system is completely characterized by its impulse response h[n] and it is possible to use above equation to compute y[n] due to any input x[n].
- And above equation is called the convolution sum.

Properties of LTI System

- **Commutative Property:**
  - Convolution operation is commutative:

$$x[n]*h[n]=h[n]*x[n]$$

- **Distributive Property:**
  - Convolution operation also distributes over addition:

$$x[n]*\left(h_1[n]+h_2[n]\right)=x[n]*h_1[n]+x[n]*h_2[n]$$

- **Associative Property:**

$$x[n]*\left(y[n]*z[n]\right)=\left(x[n]*y[n]\right)*z[n]$$

Properties of LTI System (cont.)

- **Identity:** $x[n]*\delta[n]=x[n]$
- **Delay Operation:** $x[n]*\delta[n-k]=x[n-k]$
- **Multiplication by a Constant:**

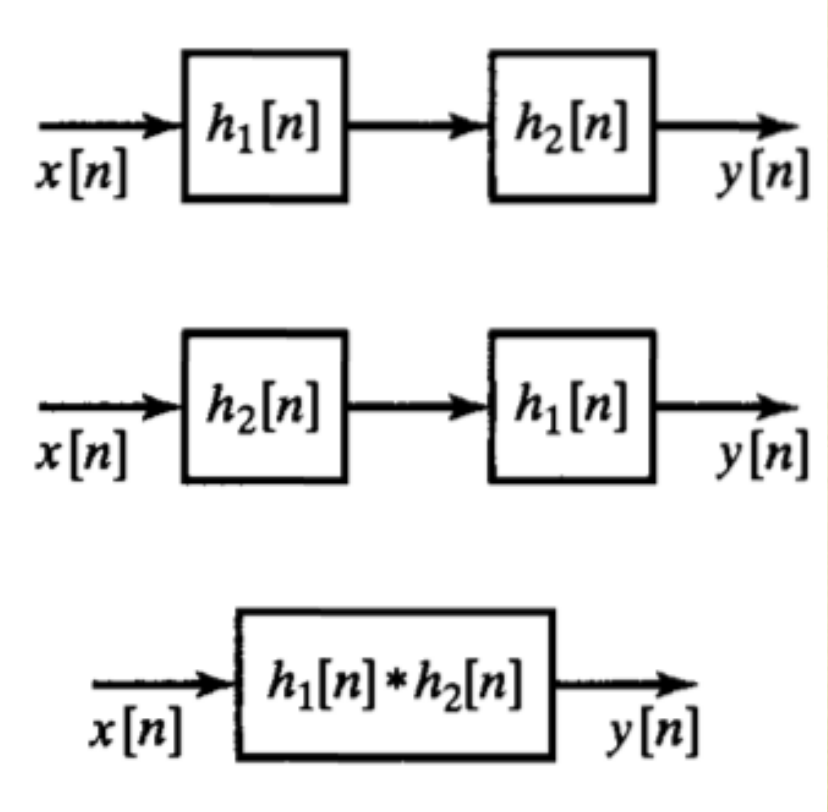
$$\{ax[n]\}*\{y[n]\}=a\left(x[n]*y[n]\right)$$

Cascading

- All above mentioned properties can be represented in cascade connection of systems.
- In a cascade connection of systems, the output of the first system is the input to the second, the output of the second is the input to the third, etc.
- The output of the last system is the overall output.

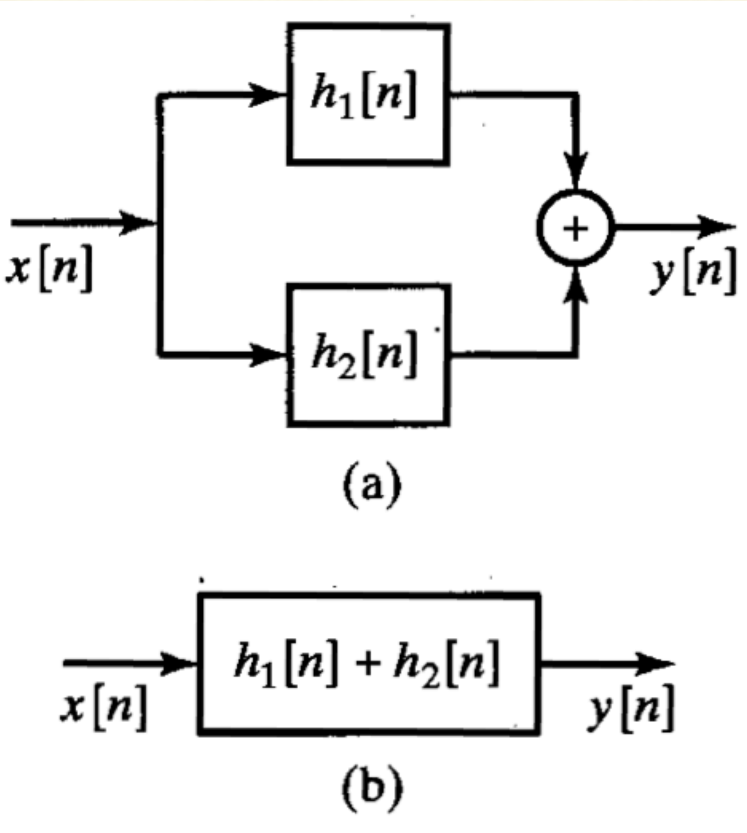
Cascading (cont.)

- **Commutative Property:**



Cascading (cont.)

- **Distributive Property:**



Cascading (cont.)

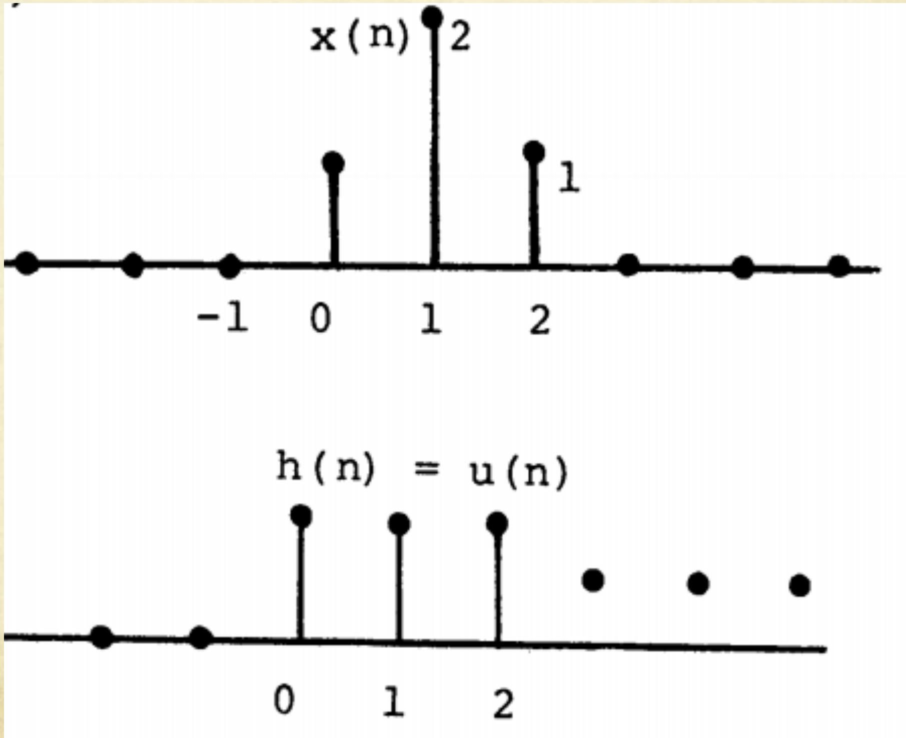
- In a parallel connection, the systems have the same input and their outputs are summed to produce and overall output.
- It follows from the distributive property of convolution that the connection of two LTI systems in parallel is equivalent to a single system whose impulse response is the sum of the individual impulse responses; i.e., $h[n]=h_1[n]+h_2[n]$

Linear Constant-Coefficient Difference Equations

- An important subclass of LTI system consists of those system for which the input and the output satisfy an N-th order linear constant-coefficient difference equation of the form: $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

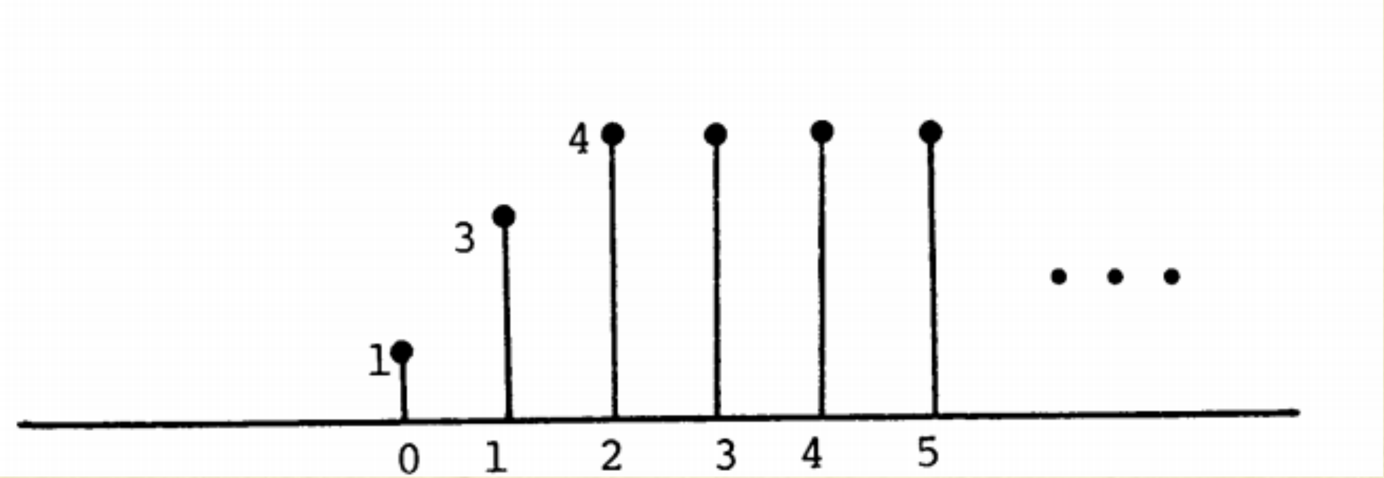
Example #2

- For the given pairs of sequences determine the output with the help of convolution:



Example #2 (cont.)

- Solution:



Example #3

- Consider a casual system for which the input x[n] and the output y[n] are related by the linear constant coefficient difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{2}x[n-1]$$

a) Determine the unit sample response of the system.

b) Using your result and the convolution sum determine the response to the input: $x[n]=e^{j\omega n}$

Example #3 (cont.)

- Solution:
  a) Rewriting the difference equation with x[n]= δ [n] and h[n] denoting the unit sample response:

$$h[n]=\frac{1}{2}h[n-1]+\delta[n]+\frac{1}{2}\delta[n-1]$$

- Since the system is casual, h(n) is zero for n<0. For n≥0,

$$h\left(0\right)=\frac{1}{2}h[-1]+\delta[0]+\frac{1}{2}\delta[-1]=1$$

$$h[1]=\frac{1}{2}h[0]+\delta[1]+\frac{1}{2}\delta[0]=1$$

$$h[2]=\frac{1}{2}h[1]+\delta[2]+\frac{1}{2}\delta[1]=\frac{1}{2}$$

$$h[n]=2\left(\frac{1}{2}\right)^{n}\quad n\geq 1$$

Example #3 (cont.)

- h[n] can also be expressed as:

$$h[n] = \left(\frac{1}{2}\right)^n \left[u(n) + u(n-1)\right]$$

b) Substituting x[n] and h[n] into the convolution sum we obtain:

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \left[u(k) + u(k-1)\right] e^{j\omega(n-k)}$$

$$= e^{j\omega n}\left[\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k e^{-j\omega k} + \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k e^{-j\omega k}\right]$$

$$= e^{j\omega n}\left[\frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right]$$

Example #4

- y(n) – 0.4 y(n-1) = x(n). Find the casual impulse response? h(n)=0 n<0.

- Solution:

$$h[n] = 0.4h[n-1] + \delta[n]$$

$$h[0] = 0.4h[-1] + \delta[0] = 1$$

$$h[1] = 0.4h[0] + \delta[1] = 0.4$$

$$h[2] = 0.4^2$$

$$h[n] = 0.4^n \quad for \quad n \geq 0$$