Digital Signal Processing
Fourier Transform-I

Lecture-3
22-March-16

Discrete Time Fourier Transform

- A discrete time signal can be represented in the frequency domain using *z*-transform or discrete time Fourier transform.
- The Fourier transform of a discrete time signal is called discrete time Fourier transform (DTFT).
- If x[n] is the given discrete time sequence then X(ω) or X($e^{j\omega}$) is the discrete time Fourier transform of x(n). i.e.,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

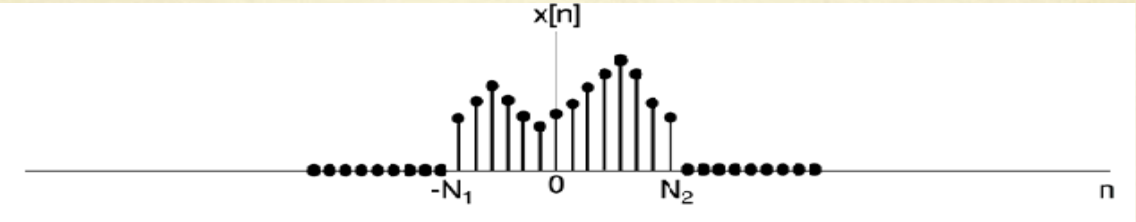
Discrete Time Fourier Transform (cont.)

- The inverse DTFT of $X(e^{j\omega})$ is defined as :

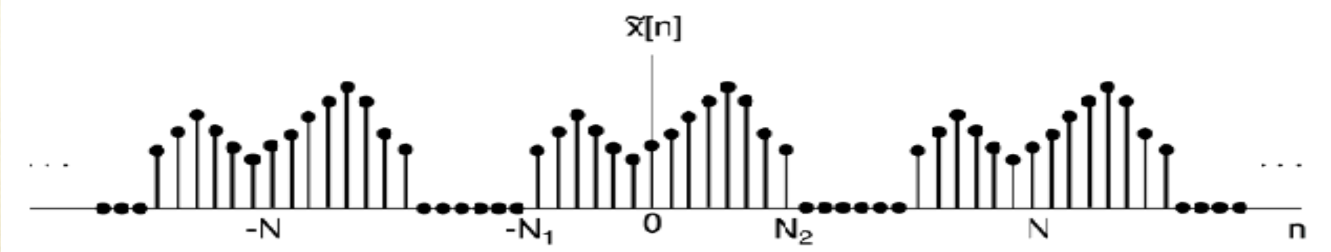
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

Derivation of DTFT

- $x[n]$ is aperiodic and of finite duration.
- $x[n]=0$ if $|n| \geq N/2$
  - N is large enough:



- $x'[n]=x[n]$ for $|n| \leq N/2$ and periodic with period N
  - $x'[n]=x[n]$ for any n as $N \rightarrow \infty$



DTFS Pairs

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \quad DTFS \quad Synthesis \quad eq.$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N} \quad DTFS \quad Analysis \quad eq.$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} X(e^{jk\omega_0})$$

$$where \quad X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

DTFS Pairs

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0$$

$$As \quad N \to \infty : \quad \tilde{x}[n] \to x[n] \quad for \quad every \quad n$$

$$\omega_0 \to 0, \sum \omega_0 \to \int d\omega$$

$$Thus, \quad x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

- The limit of integration is over any interval of 2π in ω.

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Periodic in ω with period 2π.

Discrete Time Fourier Transform

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- Analysis Equation (Fourier Transform)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

- Synthesis Equation (Inverse Fourier Transform)

Discrete Time Fourier Transform (cont.)

- Fourier transform of a signal in general is a complex valued function: i.e.,

$$X\left(e^{j\omega}\right) = X_R\left(e^{j\omega}\right) + jX_I\left(e^{j\omega}\right)$$

  - Where $X_R(e^{j\omega})$ is the real part of $X(e^{j\omega})$ and $X_I(e^{j\omega})$ is imaginary part of the function $X(e^{j\omega})$.

- Polar form is:

$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{\angle X\left(e^{j\omega}\right)}$$

  - Where $|X(e^{j\omega})|$ is magnitude and $<X(e^{j\omega})$ is the phase.

Existence of DTFT

- The Fourier transform exists for a discreet time sequence x[n] if and only if the sequence is absolutely summable.
  - The sequence has to satisfy the condition:

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

  - The DTFT does not exist for the sequences that are growing exponentially (e.g. $a^n u(n)$, $a>1$) since they are not absolutely summable.
  - This method can be applied only to stable systems and not for unstable systems.
  - That is DTFT can be used only for systems whose function H(z) has poles inside the unit circle.

Difference b/w Discrete Time and Analog Signal

- The difference b/w the Fourier transform of a discrete time signal and analog signal are as follows:

Analog Signals	Discrete Time Signal
Frequency Range : $-\infty$ to ∞	Frequency Range: $-\pi$ to π (or equivalently 0 to 2π)
It involves integration.	It involves summation.

Example #1

- If we put analysis equation into the synthesis equation we indeed get {x[n]}:
- Let: $$\hat{x}[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty}x[m]e^{-j\omega m}\right)e^{j\omega n}\,d\omega$$
  - Under the assumption that sequence {x[n]} is absolutely summable we can interchange the order of integration and summation. Thus:

$$\hat{x}[n]=\sum_{m=-\infty}^{\infty}x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{+j\omega(n-m)}\,d\omega\right)$$

$$If \quad m=n \quad then \quad e^{+j\omega(n-m)}=1$$

$$and \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}1\,d\omega=1$$

Example #1 (cont.)

$$If \quad m \neq n \quad then \quad e^{+j\omega(n-m)} = \cos\omega(n-m) + j\sin\omega(n-m)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{+j\omega(n-m)}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\omega(n-m)\, d\omega + \frac{j}{2\pi}\int_{-\pi}^{\pi} \sin\omega(n-m)\, d\omega$$

$$= \frac{1}{2\pi}\frac{\sin\omega(n-m)}{(n-m)}\Bigg|_{-\pi}^{\pi} + \frac{j}{2\pi}\frac{\cos\omega(n-m)}{(n-m)}\Bigg|_{-\pi}^{\pi} = 0$$

- Thus there is only one non zero term in R.H.S so we get x^[n]=x[n].
- This is true for all values of n and so synthesis equation is indeed a representation of signal in terms of Eigen functions {$e^{j\omega n}$).

Example #2

- Let $\{x[n]\} = \{a^n u[n]\}$, Fourier transform of this equation will exist if it is absolutely summable. We have:

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n$$

- This is a geometric series and sum exists if $|a|<1$, in that case:

$$\sum |a|^n = \frac{1}{1-|a|} < +\infty$$

- Thus the Fourier transform of the sequence $\{a^n u[n]\}$ exists if $|a|<1$. The Fourier transform is:

Example #2 (cont.)

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{n}$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

- It exists if $|ae^{-j\omega n}|<1$ i.e., $|a|<1$.

Convergence

- Absolute summability is a sufficient condition for the existence of a Fourier transform. Fourier transform also exists for square summable sequence:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- For such signals the convergence is not uniform.
- This has implications in the design of discrete system for filtering.

Properties of Impulse Function

- The impulse function is defined by the following properties:

$$(a)\quad \int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1$$

$$(b)\quad \int_{-\infty}^{\infty} X\left(e^{j\omega}\right)\delta(\omega-\omega_0)\, d\omega = X\left(e^{j\omega_0}\right)$$

If $X(e^{j\omega})$ is continuous at $\omega = \omega_0$ (shifting or convolution property)

$$(c)\quad X\left(e^{j\omega}\right)\delta(\omega) = X\left(e^{j0}\right)\delta(\omega)$$

If $X(e^{j\omega})$ is continuous at $\omega = 0$

For Example

$$(a) \quad X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega + 2\pi k\right)$$

$$(b) \quad X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega - \omega_0 + 2\pi k\right), \quad -\pi < \omega_0 \le \pi$$

Properties of DTFT

- **<u>Periodicity of the DTFT</u>:**
  - DTFT $X(e^{j\omega})$ is a periodic function of ω with period 2π.
- **<u>Linearity of the DTFT</u>:**
  - If:

$$\{x[n]\} \leftrightarrow X\left(e^{j\omega}\right)$$

  - And:

$$\{y[n]\} \leftrightarrow Y\left(e^{j\omega}\right)$$

  - Then:

$$a\{x[n]\} + b\{y[n]\} \leftrightarrow aX\left(e^{j\omega}\right) + bY\left(e^{j\omega}\right)$$

Properties of DTFT (cont.)

- **<u>Conjugation of the Signal</u>:**
  - If: $\{x[n]\} \leftrightarrow X\left(e^{j\omega}\right)$
  - Then: $\{x^*[n]\} \leftrightarrow X^*\left(e^{-j\omega}\right)$
  - Where * denotes the complex conjugate. We have DTFT of {x*[n]}:

$$\sum_{n=-\infty}^{\infty} x^*[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[x[n] e^{j\omega n}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j(-\omega)n}\right]^*$$

$$= X^*\left(e^{-j\omega}\right)$$

Properties of DTFT (cont.)

- **Time Reversal**:

$$\{x[-n]\} \leftrightarrow X\left(e^{-j\omega}\right)$$

  - The DTFT of the time reversal sequence is:

$$\sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$$

  - Let us change the index of summation as m= -n:

$$= \sum_{m=-\infty}^{\infty} x[m]e^{j\omega m} = X\left(e^{-j\omega}\right)$$

Properties of DTFT (cont.)

- **<u>Time Shifting & Frequency Shifting</u>:**

$$\left\{x\left[n-n_0\right]\right\} \leftrightarrow e^{-j\omega n_0} X\left(e^{j\omega}\right)$$

$$\left\{e^{j\omega_0 n} x\left[n\right]\right\} \leftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

- **<u>Symmetry Properties of the Fourier Transform</u>:**
  - If x[n] is real valued than:

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

  - If x[n] = x*[n], so {x[n]} = {x*[n]} and hence:

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

  - Expressing $X(e^{j\omega})$ in real and imaginary parts we see that:

$$X_R\left(e^{j\omega}\right) + jX_I\left(e^{j\omega}\right) = X_R\left(e^{-j\omega}\right) - jX_I\left(e^{-j\omega}\right)$$

Properties of DTFT (cont.)

- <u>**Symmetry Properties of the Fourier Transform**</u>: (cont.)
  - Which implies: $X_R\left(e^{j\omega}\right) = X_R\left(e^{-j\omega}\right)$

    $$and \quad X_I\left(e^{j\omega}\right) = -X_I\left(e^{-j\omega}\right)$$

  - That is real part of the Fourier transform is an even function of ω and imaginary part is an odd function of ω.
  - The magnitude spectrum is given by:

$$\left|X\left(e^{j\omega}\right)\right| = \sqrt{X_R^2\left(e^{j\omega}\right) + X_I^2\left(e^{j\omega}\right)} = \sqrt{X_R^2\left(e^{-j\omega}\right) + X_I^2\left(e^{-j\omega}\right)} = \left|X\left(e^{-j\omega}\right)\right|$$

Properties of DTFT (cont.)

- **<u>Symmetry Properties of the Fourier Transform</u>**: (cont.)
  - The phase spectrum is given by:

$$\angle X\left(e^{j\omega}\right) = \tan^{-1}\frac{X_I\left(e^{j\omega}\right)}{X_R\left(e^{j\omega}\right)}$$

$$= \tan^{-1}\frac{-X_I\left(e^{-j\omega}\right)}{X_R\left(e^{-j\omega}\right)}$$

$$= -\tan^{-1}\frac{X_I\left(e^{-j\omega}\right)}{X_R\left(e^{-j\omega}\right)}$$

$$= -\angle X\left(e^{-j\omega}\right)$$

Properties of DTFT (cont.)

- **Differencing & Summation**:

$$\{x[n]-x[n-1]\} \leftrightarrow \left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

- **Differentiation in Frequency Domain**:

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - Differentiating both sides with respect to ω, we obtain

$$\frac{d}{d\omega}X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$$

  - Multiplying both sides by j we obtain

$$\{nx[n]\} \leftrightarrow j\frac{d}{d\omega}X\left(e^{j\omega}\right)$$

Properties of DTFT (cont.)

- **Parseval's Relation**:

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

  - We have:

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(e^{j\omega}\right)e^{j\omega n}\, d\omega\right]^*$$

  - Interchanging summation and integration we get:

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*\left(e^{j\omega}\right)\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*\left(e^{j\omega}\right)X\left(e^{j\omega}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$