Digital Signal Processing
Fourier Transform-II

Lecture-4
29-March-16

Properties of DTFT (cont.)

- **<u>Convolution Property</u>**:
  - This is the Eigen function property of the complex exponential.
  - The Fourier synthesis equation for x[n] can be interpreted as a representation of {x[n]} in terms of linear combinations of complex exponential with amplitude proportional to $X(e^{j\omega})$.
  - Each of these complex exponential is an Eigen function of the LTI system and so the amplitude $Y(e^{j\omega})$in the decomposition of {y[n} will be $X(e^{j\omega})$ $H(e^{j\omega})$, where $H(e^{j\omega})$ if the Fourier transform f the impulse response.
  - <u>Prove:</u>
  - The output {y[n]} is given in terms of the convolution sum so,

$$Y\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

Properties of DTFT (cont.)

- **Convolution Property**: (cont.)

$$= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right)e^{-j\omega n}$$

Interchanging order of the summation

$$= \sum_{k=-\infty}^{\infty} h[k]\sum_{n=-\infty}^{\infty} x[n-k]e^{-j\omega n}$$

- Let m=n-k then n=m+k and we get,

$$= \sum_{k=-\infty}^{\infty} h[k]\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}$$

$$= H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

Properties of DTFT (cont.)

- **<u>Convolution Property</u>**: (cont.)
  - Thus if {y[n]}={h[n]} * {x[n]}
  - Then, $Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$
  - Convolution in time domain becomes multiplication in the frequency domain.
  - The Fourier transform of the impulse response {h[n]} is known as frequency response of the system.

Properties of DTFT (cont.)

- **<u>The Modulation or Windowing Property</u>:**
  - Let us find the DTFT of product of two sequences:

$$\{x[n]\}\{y[n]\} = \{x[n]y[n]\} = \{z[n]\}$$

$$Z\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j\omega n}$$

  - Substituting for x[n] in terms of IDFT we get,

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\alpha}\right)e^{-j\alpha n}\,d\alpha\right)y[n]e^{-j\omega n}$$

Properties of DTFT (cont.)

- **The Modulation or Windowing Property**:
  - Interchanging order of integration and summation:

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\alpha}\right)\left[\sum_{n=-\infty}^{\infty} y[n]e^{-j(\omega-\alpha)n}\right]d\alpha$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\alpha}\right)Y\left(e^{j(\omega-\alpha)}\right)d\alpha$$

  - This is the convolution of two functions only the interval of integration is $-\pi$ to π.
  - $Y(e^{j\omega})$ and $X(e^{j\omega})$ one periodic functions and equation is called the periodic convolution.

$$\{x[n]y[n]\} \leftrightarrow \frac{1}{2\pi}X\left(e^{j\omega}\right)\otimes Y\left(e^{j\omega}\right)$$

$$\textit{where} \quad \otimes \quad \textit{denotes} \quad \textit{periodic} \quad \textit{convolution}$$

Properties of DTFT

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$

Properties of DTFT (cont.)

6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

DTFT Common Pairs

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$

DTFT Common Pairs (cont.)

7. $\dfrac{r^n \sin \omega_p (n+1)}{\sin \omega_p} u[n] \quad (\lvert r \rvert < 1)$	$\dfrac{1}{1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi \delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k)]$

Systems Characterized by Linear Constant Coefficient Difference Equation

- Constant coefficient linear difference equation with zero initial condition can be used to describe some linear time invariant systems.

- The input-output {x[n]} and {y[x]} are related by:

$$\sum_{k=0}^{n} a_k y[n-k] = \sum_{k=0}^{m} x[n-k]$$

- We assume that the Fourier transform of {x[n]} , {y[n]} and {h[n]} exist, then convolution property implies that:

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

Systems Characterized by Linear Constant Coefficient Difference Equation (cont.)

- Taking Fourier transform of both sides and using linearity and time shifting property of the Fourier transform we get:

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y\left(e^{j\omega}\right) = \sum_{k=0}^{M} b_k e^{-j\omega k} X\left(e^{j\omega}\right)$$

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k} X\left(e^{j\omega}\right)}{\sum_{k=0}^{N} a_k e^{-j\omega k} Y\left(e^{j\omega}\right)}$$

Systems Characterized by Linear Constant Coefficient Difference Equation (cont.)

- Thus we see that the frequency response is ratio of polynomials in the variable $e^{-j\omega}$.

Example #1

- Determine the impulse response for a difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]-\frac{1}{4}x[n-1]$$

- Solution:

$$h[n]=\left(\frac{1}{2}\right)^{n}u[n]-\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

Example #2

- Find the DTFT of the following sequences:

$$(a) \quad \delta(n)$$

$$(b) \quad \delta(n-m)$$

Example #2 (cont.)

- Solution:

$$(a) \quad \delta(n)$$

$$X\left(e^{j\omega}\right)1$$

$$(b) \quad \delta(n-m)$$

$$X\left(e^{j\omega}\right) = e^{-j\omega m}$$

Example #3

- Find the DTFT of the following sequences:

$$(a)\quad \sin\left(\frac{n\pi}{2}\right)u(n)$$

$$(b)\quad \cos(\omega_0 n)u(n)$$

- Solution:

$$(a)\quad X\left(e^{j\omega}\right) = \frac{e^{-j\omega}}{1+e^{-j2\omega}}$$

$$(b)\quad X\left(e^{j\omega}\right) = \frac{1-e^{-j\omega}\cos\omega_0}{1-2e^{-j\omega}\cos\omega_0+e^{-j2\omega}}$$

Example #4

- Determine the signal x[n] for the given Fourier transform:

$$X\left(e^{j\omega}\right) = e^{-j\omega} \quad for \quad -\pi \le \omega \le \pi$$

- Solution:

$$x[n] = \frac{\sin \pi (n-1)}{\pi (n-1)}$$

Example #5

- Consider the linear shift invariant system characterized by the second order linear constant coefficient difference equation:

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

- Find the frequency response:
- Solution:

$$H\left(e^{j\omega}\right) = \frac{1 - 1.4142e^{-j\omega} + e^{-2j\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-2j\omega}}$$

Example #6

- Solve the following LCCDE for y(n) assuming zero initial conditions and x[n]= δ (n):

$$y[n]-0.25y[n-1]=x[n]-x[n-2]$$

- Solution:

$$y[n]=(0.25)^{n}u(n)-(0.25)^{n-2}u(n-2)$$