Digital Signal Processing
Z-Transform-I

Lecture-5
30-March-16

Introduction

- Z-transform is a useful tool in the analysis of discrete-time signals and systems.
- It is the discrete time counterpart of the Laplace transform for continuous-time signal and systems.
- It can be used to solve constant coefficient difference equations, evaluate the response of a linear time-invariant system to a given input and design linear filters.

Z-Transform

- The *z*-transform of a sequence x[n] is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \rightarrow eq(1)$$

  - Where $z=re^{j\omega}$ is a complex variable.

- Above equation can be considered as an operator that transforms a sequence into a function, and we will refer to the *z*-transform operator Z(.) defined as:

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

- The values of *z* for which the sum converges define a region in the *z*-plane referred to as region of convergence (ROC).

Z-Transform

- The correspondence between a sequence and its *z*-transform is indicated by the notation:

$$x[n] \overset{Z}{\leftrightarrow} X(z)$$

- The *z*-transform defined in eq.(1) is referred to as the two-sided or bilateral *z*-transform, in contrast to the one sided or unilateral *z*-transform which is defined as:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Relation b/w Z-Transform & DTFT

- There is a close relation between the two.
- If complex variable *z* is replaced with the complex variable $e^{j\omega}$, then the *z*-transform reduces to the Fourier transform.
- When Fourier transform exists, the Fourier transform is simple X(z) with $z=e^{j\omega}$.
- This corresponds to restricting *z* to have unity magnitude i.e., for $|z|=1$, the *z*-transform corresponds to the Fourier transform.
- The complex variable *z* in polar form can be expressed as: $z=re^{j\omega}$.

Relation b/w Z-Transform & DTFT (cont.)

- The eq.(1) becomes:

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n}$$

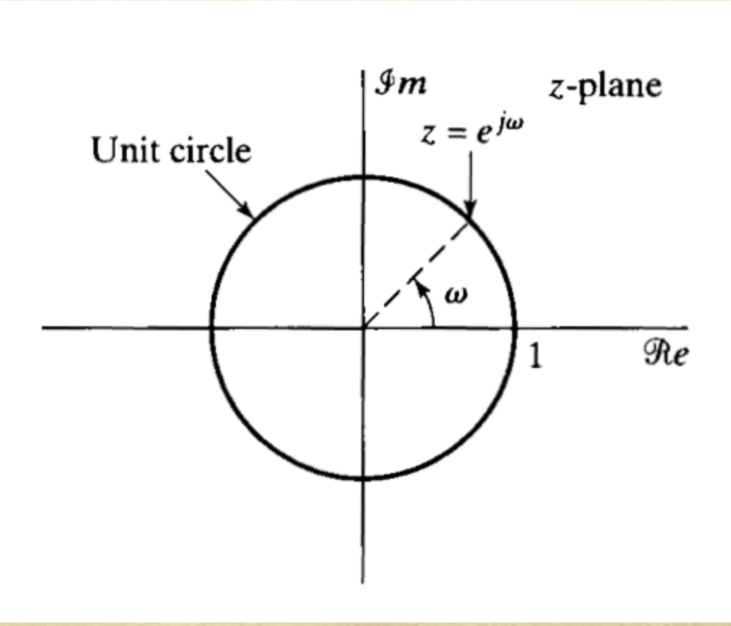
or

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right)\left(e^{-j\omega n}\right)$$

- We see that X(z) is the discrete time Fourier transform of the sequence r^{-n} x[n]. When r=1 it reduces to the Fourier transform of x[n].

Region of Convergence (ROC)

- It is convenient to interpret z transform using the complex z-plane.
- In the z-plane, the contour corresponding to $|z|=1$ is a circle of unit radius and the contour is referred to as the unit circle.



Region of Convergence (ROC) (cont.)

- ROC is determined by the range of values of e for which:

$$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty \rightarrow eq(2)$$

- And:

$$z = \mathrm{Re}(z) + j\,\mathrm{Im}(z) = re^{j\omega}$$

- The axes of *z*-plane are the real and imaginary parts of *z*.
- The *z*-transform evaluated on the unit circle corresponds to the DTFT :

$$X\left(e^{j\omega}\right) = X(z)\Big|_{z=e^{j\omega}}$$

- The set of values of *z* for which the *z*-transform converges is called the region of convergence.

Region of Convergence (ROC) (cont.)

- Evaluating X(z) at points around the unit circle, beginning at $z=1$ ($\omega=0$) through $z=j$ (i.e., $\omega=\pi/2$) to $z=-1$ (i.e., $\omega=\pi$).
- We obtain values of $X(e^{j\omega})$ for $0 \le \omega \le \pi$.
- Note: In order for the DTFT of a signal to exist, the unit circle must be within the ROC of X(z).
- In eq.(2) the multiplication of complex variable z by the real exponential r^{-n}, it is possible for the *z*-transform to converge even if the Fourier transform does not.
- X(z) is a rational function inside the ROC, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}$$

Region of Convergence (ROC) (cont.)

- Where $P(z)$ and $Q(z)$ are polynomials in z.
- The values of z for which $X(z)=0$ are called the zeroes of $X(z)$.
- The values for which $X(z)$ is infinite are referred to as the poles of $X(z)$.
- The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial.
- The poles and zeroes uniquely define the functional form of a rational z-transform to within a constant.

Properties of ROC

- Property 1:
  - The ROC is a ring or disk in the *z*-plane centered at the origin, i.e., $0 \le r_R < |z| < r_L \le \infty$.
- Property 2:
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the *z*-transform of x[n] includes the unit circle.
- Property 3:
  - The ROC cannot contain any poles.
- Property 4:
  - If x[n] is a finite duration sequence, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire *z*-plane, except possibly $z=0$ or $z=\infty$.

Properties of ROC (cont.)

- Property 5:
  - If x[n] is a right-sided sequence, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outwards from the outermost (i.e., largest magnitude) finite pole in X(z) to $z=\infty$.
- Property 6:
  - If x[n] is a left-sided sequence, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inwards from the innermost (i.e., smallest magnitude) nonzero pole in X(z) to $z=0$.

Properties of ROC (cont.)

- Property 7:
  - A two-sided sequence is an infinite duration sequence that is neither right sided nor left sided. If x[n] is a two sided sequence, the ROC will consist of a ring in the *z*-plane, bounded on the interior and exterior by a pole and consistent with property 3, not containing any poles.
- Property 8:
  - The ROC must be a connected region.

Common Z-Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

Common Z-Transform Pairs (cont.)

6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$

Common Z-Transform Pairs (cont.)

11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Example #1

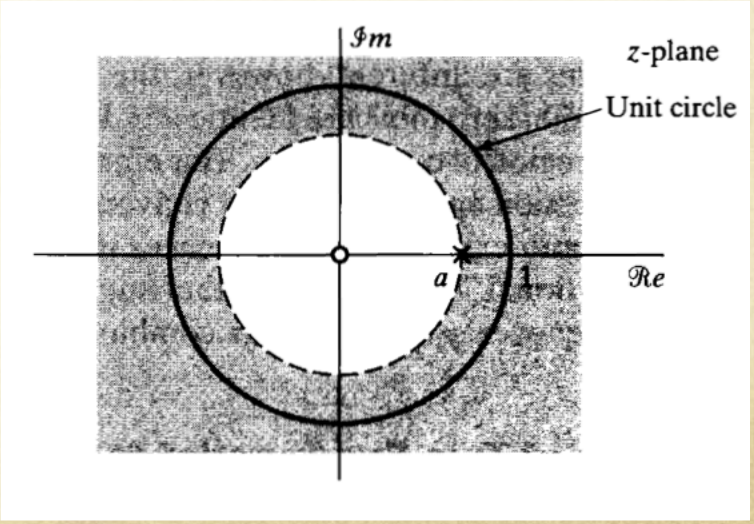
- Find the *z*-transform of the sequence:

$$x[n] = \alpha^n u[n]$$

- Solution:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$



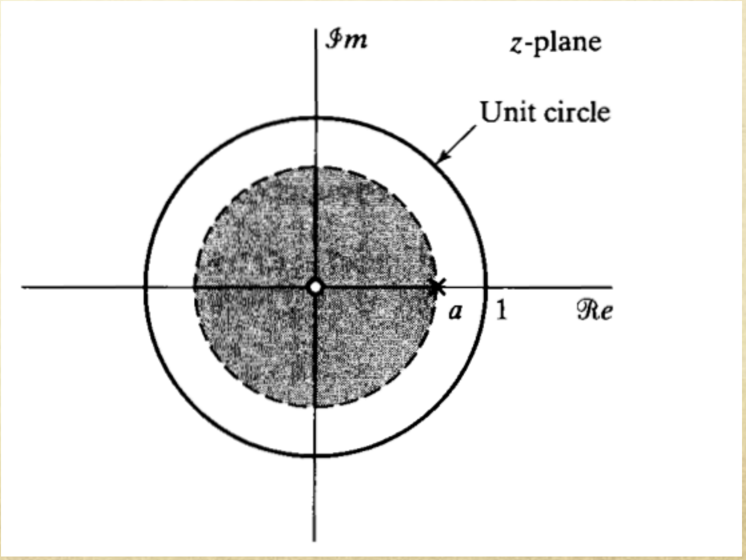
Example #2

- Find the *z*-transform of the sequence:

$$x[n] = -\alpha^n u[-n-1]$$

- Solution:

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$



Example #3

- Find the *z*-transform of :

$$x[n] = \left(\frac{1}{2}\right)^n u(n) - 2^n u(-n-1)$$

- Solution:

$$X(z) = \frac{2 - \frac{5}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$