Examples of Lecture #4:

- Example#2:

$$(a) \quad \delta(n)$$

$$(b) \quad \delta(n-m)$$

Solution:

(a) Given $x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$X(\omega) = \mathrm{F}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n} \Big|_{n=0} = 1$$

$\therefore$ $\mathrm{F}\{\delta(n)\} = 1$

$$\boxed{\delta(n) \xleftrightarrow{\text{FT}} 1}$$

(c) Given $x(n) = \delta(n - m)$

$$\delta(n-m) = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

$$X(\omega) = \mathrm{F}\{\delta(n-m)\} = \sum_{n=-\infty}^{\infty} \delta(n-m)\, e^{-j\omega n} = e^{-j\omega n}\Big|_{n=m} = e^{-j\omega m}$$

$\therefore$ $\mathrm{F}\{\delta(n-m)\} = e^{-j\omega m}$

$$\boxed{\delta(n-m) \xleftrightarrow{\text{FT}} e^{-j\omega m}}$$

- Example #3:

$$(a) \quad \sin\left(\frac{n\pi}{2}\right)u(n)$$

$$(b) \quad \cos(\omega_0 n)u(n)$$

(a) Given $$x(n) = \sin\left(\frac{n\pi}{2}\right)u(n)$$

$$X(\omega) = \mathrm{F}\left\{\sin\left(\frac{n\pi}{2}\right)u(n)\right\} = \sum_{n=-\infty}^{\infty}\left\{\sin\left(\frac{n\pi}{2}\right)u(n)\right\}e^{-j\omega n} = \sum_{n=0}^{\infty}\sin\left(\frac{n\pi}{2}\right)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\frac{e^{j(n\pi/2)} - e^{-j(n\pi/2)}}{2j}e^{-j\omega n} = \frac{1}{2j}\left[\sum_{n=0}^{\infty}e^{j[(\pi/2)-\omega]n} - \sum_{n=0}^{\infty}e^{-j[(\pi/2)+\omega]n}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j[(\pi/2)-\omega]}} - \frac{1}{1-e^{-j[(\pi/2)+\omega]}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j(\pi/2)}e^{-j\omega} - 1 + e^{j(\pi/2)}e^{-j\omega}}{1+e^{-j2\omega} - e^{-j\omega}[e^{j(\pi/2)} + e^{-j(\pi/2)}]}\right]$$

$$= \frac{e^{-j\omega}\sin(\pi/2)}{1+e^{-j2\omega} - e^{-j\omega}\cos(\pi/2)} = \frac{e^{-j\omega}}{1+e^{-j2\omega}}$$

(e) Given $$x(n) = \cos(\omega_0 n)\, u(n)$$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-\infty}^{\infty}\{\cos(\omega_0 n)\,u(n)\}\,e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right]e^{-j\omega n}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty}[e^{j(\omega_0-\omega)}]^n + \sum_{n=0}^{\infty}[e^{-j(\omega_0+\omega)}]^n\right\}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j(\omega_0-\omega)}} + \frac{1}{1-e^{-j(\omega_0+\omega)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\omega_0+\omega)} + 1 - e^{j(\omega_0-\omega)}}{1+e^{-j2\omega} - e^{-j\omega}(e^{j\omega_0} + e^{-j\omega_0})}\right]$$

$$= \frac{1-e^{-j\omega}\cos\omega_0}{1-2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}$$

- Example # 4:

$$X\left(e^{j\omega}\right) = e^{-j\omega} \quad for \quad -\pi \le \omega \le \pi$$

(a) Given $X(\omega) = e^{-j\omega}$

$$x(n) = \mathrm{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-1)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-1)}}{j(n-1)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\pi(n-1)} - e^{-j\pi(n-1)}}{j(n-1)}\right] = \frac{1}{\pi(n-1)}\left[\frac{e^{j\pi(n-1)} - e^{-j\pi(n-1)}}{2j}\right]$$

$$= \frac{\sin \pi(n-1)}{\pi(n-1)}$$

- Example #1:

$$y[n]-\frac{1}{2}y[n-1]=x[n]-\frac{1}{4}x[n-1]$$

Example 2.29 Determining the Impulse Response for a Difference Equation

In this example we determine the impulse response for a stable linear time-invariant system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] - \tfrac{1}{4}x[n-1]. \tag{2.180}$$

In Chapter 3 we will see that the z-transform is more useful than the Fourier transform for dealing with difference equations. However, this example offers a hint of the utility of transform methods in the analysis of linear systems. To find the impulse response, we set $x[n] = \delta[n]$; with $h[n]$ denoting the impulse response, Eq. (2.180) becomes

$$h[n] - \tfrac{1}{2}h[n-1] = \delta[n] - \tfrac{1}{4}\delta[n-1]. \tag{2.181}$$

Applying the Fourier transform to both sides of Eq. (2.181) and using properties 1 and 2 of Table 2.2, we obtain

$$H(e^{j\omega}) - \tfrac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \tfrac{1}{4}e^{-j\omega}, \tag{2.182}$$

or

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{2.183}$$

To obtain $h[n]$, we want to determine the inverse Fourier transform of $H(e^{j\omega})$. Toward this end, we rewrite Eq. (2.183) as

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{2.184}$$

From transform 4 of Table 2.3,

$$\left(\tfrac{1}{2}\right)^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Combining this transform with property 3 of Table 2.2, we obtain

$$-\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} -\frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{2.185}$$

Based on property 1 of Table 2.2, then,

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1]. \tag{2.186}$$

- Example #5:

$$y[n]=1.3433y[n-1]-0.9025y[n-2]+x[n]-1.4142x[n-1]+x[n-2]$$

Example 2.7

Consider the linear shift-invariant system characterized by the second-order linear constant coefficient difference equation

$$y(n) = 1.3433y(n-1) - 0.9025y(n-2) + x(n) - 1.4142x(n-1) + x(n-2)$$

The frequency response may be found by inspection without solving the difference equation for $h(n)$ as follows:

$$H(e^{j\omega}) = \frac{1 - 1.4142e^{-j\omega} + e^{-2j\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-2j\omega}}$$

Note that this problem may also be worked in the reverse direction. For example, given a frequency response function such as

$$H(e^{j\omega}) = \frac{1 + e^{-2j\omega}}{2 - e^{-j\omega} + 0.5e^{-2j\omega}}$$

a difference equation may be easily found that will implement this system. First, dividing numerator and denominator by 2 and rewriting the frequency response as follows,

$$H(e^{j\omega}) = \frac{0.5 + 0.5e^{-2j\omega}}{1 - 0.5e^{-j\omega} + 0.25e^{-2j\omega}}$$

we see that a difference equation for this system is

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + 0.5x(n) + 0.5x(n-2)$$

- Example #6:

$$y[n]-0.25y[n-1]=x[n]-x[n-2]$$

Example 2.9

Let us solve the following LCCDE for $y(n)$ assuming zero initial conditions,

$$y(n) - 0.25y(n-1) = x(n) - x(n-2)$$

for $x(n) = \delta(n)$. We begin by taking the DTFT of each term in the difference equation:

$$Y(e^{j\omega}) - 0.25e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) - e^{-2j\omega}X(e^{j\omega})$$

Because the DTFT of $x(n)$ is $X(e^{j\omega}) = 1$,

$$Y(e^{j\omega}) = \frac{1 - e^{-2j\omega}}{1 - 0.25e^{-j\omega}} = \frac{1}{1 - 0.25e^{-j\omega}} - \frac{e^{-2j\omega}}{1 - 0.25e^{-j\omega}}$$

Using the DTFT pair

$$(0.25)^n u(n) \stackrel{DTFT}{\Longleftrightarrow} \frac{1}{1 - 0.25e^{-j\omega}}$$

the inverse DTFT of $Y(e^{j\omega})$ may be easily found using the linearity and shift properties.

$$y(n) = (0.25)^n u(n) - (0.25)^{n-2}u(n-2)$$

Examples of Lecture #5:

- Example #1:

$$x[n] = \alpha^n u[n]$$

Example 3.1 Right-Sided Exponential Sequence

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the region of convergence is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the region of convergence, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \tag{3.10}$$

Here we have used the familiar formula for the sum of terms of a geometric series. The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{3.11}$$

- Example #2:

$$x[n] = -\alpha^n u[-n-1]$$

Example 3.2 Left-Sided Exponential Sequence

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \leq -1$, this is a *left-sided* sequence. Then

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n. \end{aligned} \tag{3.12}$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the sum in Eq. (3.12) converges, and

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|. \tag{3.13}$$

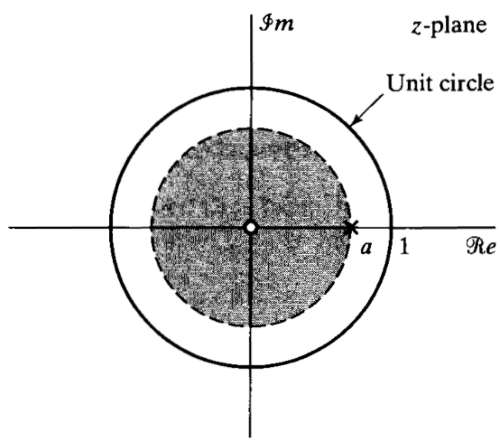


Figure 3.4 Pole–zero plot and region of convergence for Example 3.2.

- Example #3:

$$x[n]=\left(\frac{1}{2}\right)^n u(n)-2^n u(-n-1)$$

Example 4.3

Find the z-transform of $x(n) = \left(\frac{1}{2}\right)^n u(n) - 2^n u(-n-1)$, and find another signal that has the same z-transform but a different region of convergence.

Here, we have a sum of two sequences. Therefore, we may find the z-transform of each sequence separately and add them together. From Example 4.1, we know that the z-transform of $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ is

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

and from Example 4.2 that the z-transform of $x_2(n) = -2^n u(-n-1)$ is

$$X_2(z) = \frac{1}{1-2z^{-1}} \qquad |z| > 2$$

Therefore, the z-transform of $x(n) = x_1(n) + x_2(n)$ is

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} = \frac{2-\frac{5}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}$$

with a region of convergence $\frac{1}{2} < |z| < 2$, which is the set of all points that are in the ROC of both $X_1(z)$ and $X_2(z)$.

To find another sequence that has the same z-transform, note that because $X(z)$ is a sum of two z-transforms,

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}$$

each term corresponds to the z-transform of either a right-sided or a left-sided sequence, depending upon the region of convergence. Therefore, choosing the right-sided sequences for both terms, it follows that

$$x^1(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(n)$$

has the same z-transform as $x(n)$, except that the region of convergence is $|z| > 2$.

Examples of Lecture #6:

- Example #3:

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)},\quad |z|>\frac{1}{2}$$

Example 3.8 Second-Order z-Transform

Consider a sequence $x[n]$ with z-transform

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)},\qquad |z|>\frac{1}{2}. \tag{3.42}$$

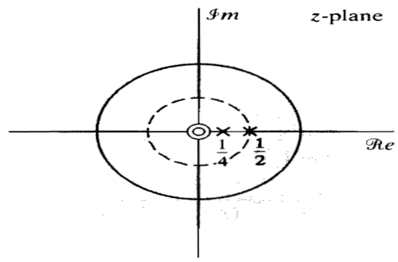


Figure 3.12 Pole-zero plot and ROC for Example 3.8.

The pole–zero plot for $X(z)$ is shown in Figure 3.12. From the region of convergence and property 5, Section 3.2, we see that $x[n]$ is a right-sided sequence. Since the poles are both first order, $X(z)$ can be expressed in the form of Eq. (3.40); i.e.,

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}.$$

From Eq. (3.41),

$$A_1=\left(1-\tfrac{1}{4}z^{-1}\right)X(z)\Big|_{z=1/4}=-1,$$

$$A_2=\left(1-\tfrac{1}{2}z^{-1}\right)X(z)\Big|_{z=1/2}=2.$$

Therefore,

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}.$$

Since $x[n]$ is right sided, the ROC for each term extends outward from the outermost pole. From Table 3.1 and the linearity of the z-transform, it then follows that

$$x[n]=2\left(\frac{1}{2}\right)^n u[n]-\left(\frac{1}{4}\right)^n u[n].$$

- Example #4:

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}, \quad |z|>1$$

Example 3.9 Inverse by Partial Fractions

To illustrate the case in which the partial fraction expansion has the form of Eq. (3.43), consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}, \qquad |z|>1. \qquad (3.46)$$

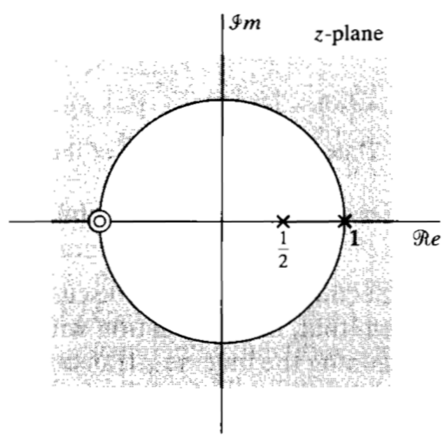


Figure 3.13 Pole–zero plot for the z-transform in Example 3.9.

The pole–zero plot for $X(z)$ is shown in Figure 3.13. From the region of convergence and property 5, Section 3.2, it is clear that $x[n]$ is a right-sided sequence. Since $M = N = 2$ and the poles are all first order, $X(z)$ can be represented as

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}.$$

The constant B_0 can be found by long division:

$$\begin{array}{r|l} & \quad\quad\quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad\quad 5z^{-1} - 1 \end{array}$$

Since the remainder after one step of long division is of degree 1 in the variable z^{-1}, it is not necessary to continue to divide. Thus, $X(z)$ can be expressed as

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}. \quad (3.47)$$

Now the coefficients A_1 and A_2 can be found by applying Eq. (3.41) to Eq. (3.46) or, equivalently, Eq. (3.47). Using Eq. (3.47), we obtain

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - z^{-1}\right)\right]_{z=1} = 8.$$

Therefore,

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}. \quad (3.48)$$

From Table 3.1, we see that since the ROC is $|z| > 1$,

$$2 \overset{\mathcal{Z}}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1 - \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n],$$

$$\frac{1}{1 - z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} u[n].$$

Thus, from the linearity of the z-transform,

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

- Example # 5:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

Example 3.10 Finite-Length Sequence

Suppose $X(z)$ is given in the form

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}). \tag{3.50}$$

Although $X(z)$ is obviously a rational function, its only poles are at $z = 0$, so a partial fraction expansion according to the technique of Section 3.3.2 is not appropriate. However, by multiplying the factors of Eq. (3.50), we can express $X(z)$ as

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}.$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

- Example #6:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Example 3.12 Power Series Expansion by Long Division

Consider the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{3.53}$$

Since the region of convergence is the exterior of a circle, the sequence is a right-sided one. Furthermore, since $X(z)$ approaches a finite constant as z approaches infinity, the sequence is causal. Thus, we divide, so as to obtain a series in powers of z^{-1}. Carrying out the long division, we obtain

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \quad \cdots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots.$$

Hence, $x[n] = a^n u[n]$.

- Example #7:

$$X(z)=\frac{z+2z^{-2}+z^{-3}}{1-3z^{-4}+z^{-5}}$$

4.1 The z-transform of a sequence $x(n)$ is

$$X(z)=\frac{z+2z^{-2}+z^{-3}}{1-3z^{-4}+z^{-5}}$$

If the region of convergence includes the unit circle, find the DTFT of $x(n)$ at $\omega=\pi$.

If $X(z)$ is the z-transform of $x(n)$, and the unit circle is within the region of convergence, the DTFT of $x(n)$ may be found by evaluating $X(z)$ around the unit circle:

$$X(e^{j\omega})=X(z)|_{z=-1}$$

Therefore, the DTFT at $\omega=\pi$ is

$$X(e^{j\omega})|_{\omega=\pi}=X(z)|_{z=e^{j\pi}}=X(z)|_{z=-1}$$

and we have

$$X(e^{j\omega})|_{\omega=\pi}=\left.\frac{z+2z^{-2}+z^{-3}}{1-3z^{-4}+z^{-5}}\right|_{z=e^{j\pi}}=\frac{-1+2-1}{1-3-1}=0$$

- Example #8:

$$(a)\quad x[n]=\cos(n\omega_0)u(n)$$

$$(b)\quad x[n]=\left(\frac{1}{3}\right)^n u(-n)$$

$$(c)\quad x[n]=\left(\frac{1}{2}\right)^n u(n+2)+(3)^n u(-n-1)$$

(b) For this sequence we write

$$x(n)=\cos(n\omega_0)u(n)=\frac{1}{2}[e^{jn\omega_0}+e^{-jn\omega_0}]u(n)$$

Therefore, the z-transform is

$$X(z)=\frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}}+\frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}$$

with a region of convergence $|z|>1$. Combining the two terms together, we have

$$X(z)=\frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}\qquad |z|>1$$

4 Find the z-transform of each of the ...

4.4 Find the z-transform of each of the following sequences. Whenever convenient, use the properties of the z-transform to make the solution easier.

(a) $x(n) = \left(\frac{1}{3}\right)^n u(-n)$ **(b)** $x(n) = \left(\frac{1}{2}\right)^n u(n+2) + (3)^n u(-n-1)$

(c) $x(n) = \left(\frac{1}{3}\right)^n \cos(n\omega_0)u(n)$ **(d)** $x(n) = \alpha^{|n|}$

(a) Using the definition of the z-transform we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} 3^n z^n = \frac{1}{1-3z}$$

where the sum converges for

$$|3z| < 1 \text{ or } |z| < \frac{1}{3}$$

Alternatively, note that the time-reversed sequence $y(n) = x(-n) = \left(\frac{1}{3}\right)^{-n} u(n)$ has a z-transform given by

$$Y(z) = \frac{1}{1-3z^{-1}}$$

with a region of convergence given by $|z| > 3$. Therefore, using the time-reversal property, $Y(z) = X(z^{-1})$, we obtain the same result.

(b) Because $x(n)$ is the sum of two sequences, we will find the z-transform of $x(n)$ by finding the z-transforms of each of these sequences and adding them together. The z-transform of the first sequence may be found easily using the shift property. Specifically, note that because

$$\left(\frac{1}{2}\right)^n u(n+2) = 4\left(\frac{1}{2}\right)^{n+2} u(n+2)$$

the z-transform of $\left(\frac{1}{2}\right)^n u(n+2)$ is $4z^2$ times the z-transform of $\left(\frac{1}{2}\right)^n u(n)$, that is,

$$\left(\frac{1}{2}\right)^n u(n+2) \stackrel{z}{\longleftrightarrow} \frac{4z^2}{1-\frac{1}{2}z^{-1}}$$

which has a region of convergence $|z| > \frac{1}{2}$.

The second term is a left-sided exponential and has a z-transform that we have seen before, that is,

$$3^n u(-n-1) \stackrel{z}{\longleftrightarrow} -\frac{1}{1-3z^{-1}}$$

with a region of convergence $|z| < 3$.

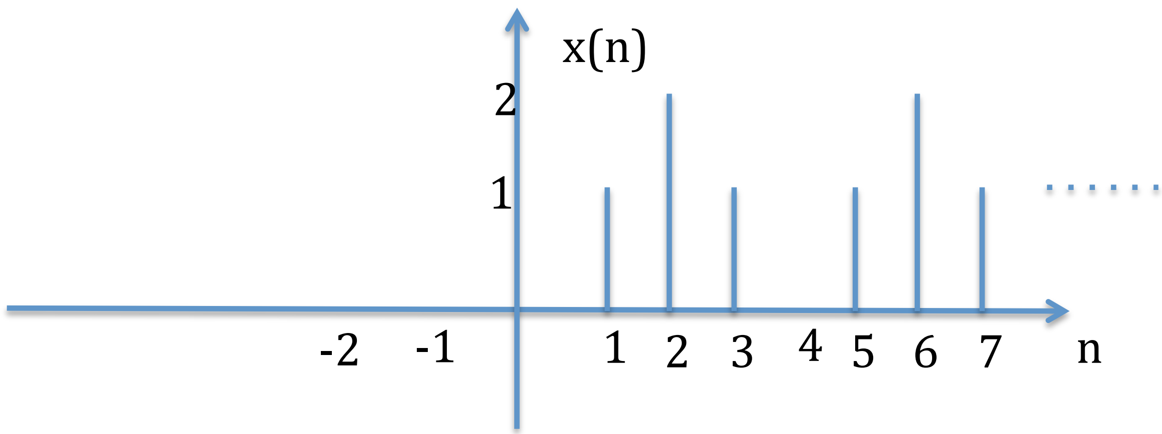
Finally, for the z-transform of $x(n)$, we have

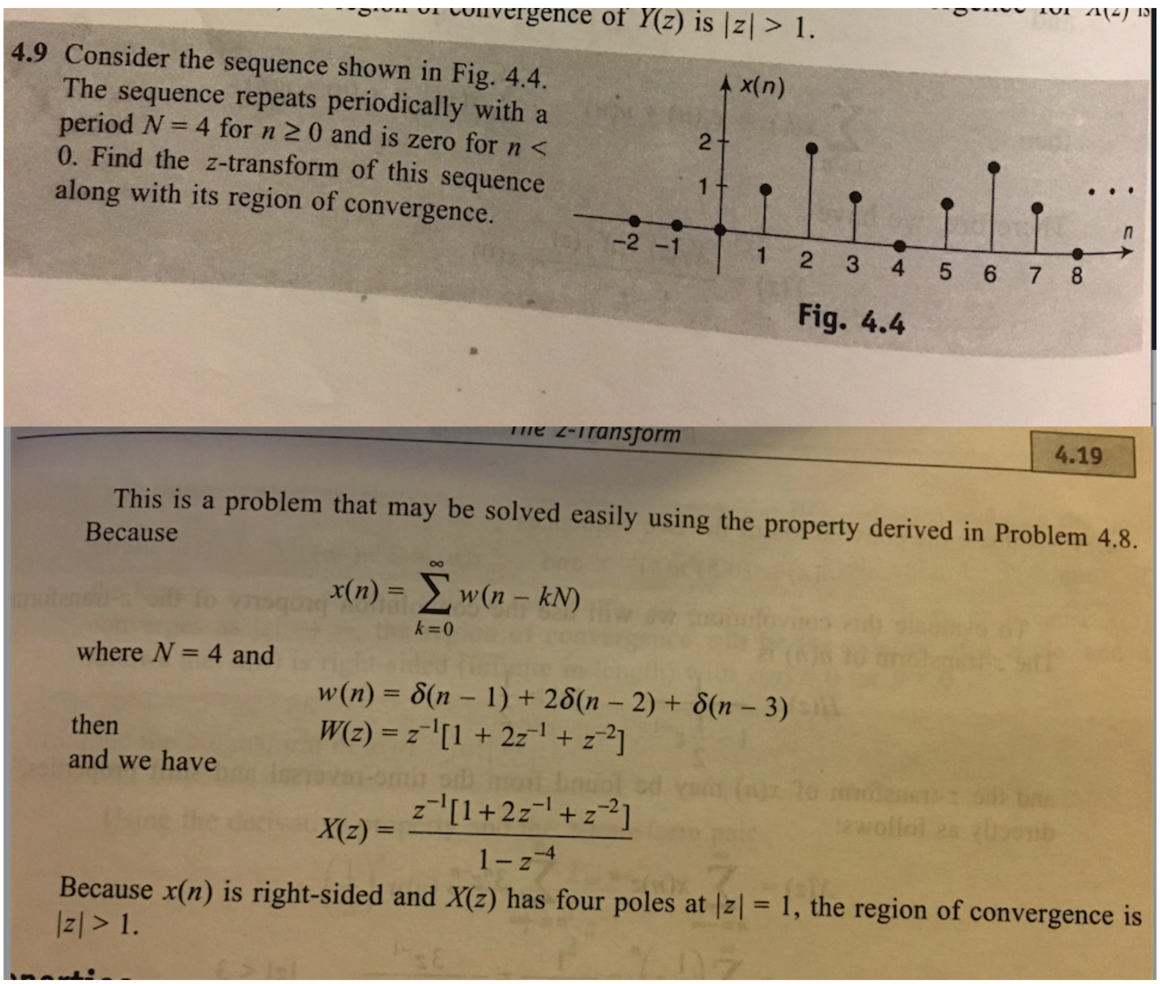
$$X(z) = \frac{4z^2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-3z^{-1}}$$

with a region of convergence $\frac{1}{2} < |z| < 3$.

...transform of $\cos(n\omega_0)u(n)$ is

- Example #9:





...region of convergence of $Y(z)$ is $|z| > 1$.

4.9 Consider the sequence shown in Fig. 4.4. The sequence repeats periodically with a period $N = 4$ for $n \geq 0$ and is zero for $n < 0$. Find the z-transform of this sequence along with its region of convergence.

Fig. 4.4

This is a problem that may be solved easily using the property derived in Problem 4.8. Because

$$x(n) = \sum_{k=0}^{\infty} w(n - kN)$$

where $N = 4$ and

$$w(n) = \delta(n-1) + 2\delta(n-2) + \delta(n-3)$$

then

$$W(z) = z^{-1}[1 + 2z^{-1} + z^{-2}]$$

and we have

$$X(z) = \frac{z^{-1}[1 + 2z^{-1} + z^{-2}]}{1 - z^{-4}}$$

Because $x(n)$ is right-sided and $X(z)$ has four poles at $|z| = 1$, the region of convergence is $|z| > 1$.

- Example #10:

$$r_x(n) = \sum_{k=-\infty}^{\infty} x(k)x(n+k)$$

$$y(120) = 4{,}449.56$$

4.39 The *deterministic autocorrelation sequence* corresponding to a sequence $x(n)$ is defined as

$$r_x(n) = \sum_{k=-\infty}^{\infty} x(k)x(n+k)$$

(a) Express $r_x(n)$ as the convolution of two sequences, and find the z-transform of $r_x(n)$ in tenns of the z-transform of $x(n)$.

(b) If $x(n) = a^n u(n)$, where $|a| < 1$, find the autocorrelation sequence, $r_x(n)$, and its z-transform.

(a) From the definition of the deterministic autocorrelation, we see that $r_x(n)$ is the convolution of $x(n)$ with $x(-n)$,

$$r_x(n) = x(n) * x(-n)$$

Therefore, using the time-reversal property of the z-transform, it follows that

$$R_x(z) = X(z)X(z^{-1})$$

If the region of convergence of $X(z)$ is R_x, the region of convergence of $R_x(z)$ will be the intersection of the regions R_x and $1/R_x$. Therefore, if this intersection is to be nonempty, R_x must include the unit circle.

(b) With $x(n) = a^n u(n)$, the z-transform is

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

and the z-transform of the autocorrelation sequence is

$$R_x(z) = \frac{1}{(1 - az^{-1})(1 - az)} \qquad |a| < |z| < \frac{1}{|a|}$$

The autocorrelation sequence may be found by computing the inverse z-transform of $R_x(z)$. Performing a partial fraction expansion of $R_x(z)$, we have

$$R_x(z) = \frac{1}{(1 - az^{-1})(1 - az)} = \frac{-a^{-1}z^{-1}}{(1 - az^{-1})(1 - a^{-1}z^{-1})} = \frac{1}{1 - a^2}\left[\frac{1}{1 - az^{-1}} - \frac{1}{1 - a^{-1}z^{-1}}\right]$$

Thus; because the region of convergence is $|a| < z < 1/|a|$, the inverse z-transform is

$$r_x(n) = \frac{1}{1 - a^2}[a^n u(n) + a^{-n} u(-n-1)] = \frac{1}{1 - a^2} a^{|n|}$$

many disciplines, differential equations play a major role in characterizing the behavior

- Example #11:

$$x(n) = \left(\frac{1}{3}\right)^3 u(n+3)$$

4.30 Find the one-sided z-transform of the following sequences:

(a) $x(n) = \left(\frac{1}{3}\right)^3 u(n+3)$ **(b)** $x(n) = \delta(n-5) + \delta(n) + 2^{n-1}u(-n)$

In the following, let $x_+(n)$ denote the sequence that is formed from $x(n)$ by setting $x(n)$ equal to zero for $n < 0$, that is,

$$x_+(n) = \begin{cases} x(n) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(a) Because $x_+(n) = \left(\frac{1}{3}\right)^n u(n)$, the one-sided z-transform of $x(n)$ is

$$X_1(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

(b) For this sequence because