Digital Signal Processing
Z-Transform-II

Lecture-6
05-April-16

Properties of Z-Transform

- **Linearity:**
  - Z-transform is a linear operator.
  - If x[n] has a *z*-transform X(z) with a region of convergence R_x, and y[n] has a *z*-transform Y(z) with a region of convergence R_y, then:

$$w[n] = ax[n] + by[n] \overset{z}{\leftrightarrow} W[z] = aX[z] + bY[z]$$

  - The ROC of w[n] will include the intersection of R_x and R_y, that is:

$$R_w \quad contains \quad R_x \cap R_y$$

Properties of Z-Transform (cont.)

- **Shifting Property:**
  - Shifting a sequence (delaying or advancing) multiplies the z-transform by a power of z.
  - If x[n] has a z-transform X(z):

$$x[n-n_0] \overset{Z}{\leftrightarrow} z^{-n_0} X[z]$$

  - The quantity n_0 is an integer.
  - If n_0 is positive, the original sequence x[n] is shifted right and if n_0 is negative, x[n] is shifted left.

Properties of Z-Transform (cont.)

- **Time Reversal:**
  - If x[n] has a *z*-transform X(*z*) with a region of convergence R_x, that is the annulus $\alpha < |z| < \beta$, the *z*-transform of the time reversal sequence x(-n) is:

$$x(-n) \overset{Z}{\leftrightarrow} X\left(z^{-1}\right)$$

  - And has a region of convergence $1/\beta < |z| < 1/\alpha$, which is denoted by $1/R_x$.

Properties of Z-Transform (cont.)

- **Multiplication by an Exponential:**
  - If a sequence x(n) is multiplied by a complex exponential α^n.

$$\alpha^n x(n) \overset{Z}{\leftrightarrow} X\left(\alpha^{-1} z\right)$$

  - If the ROC of X(z) is $r_- < |z| < r_+$, which will be denoted by R_x, the ROC of $X(\alpha^{-1}z)$ is $|\alpha| r_- < |z| < |\alpha| r_+$ which is denoted by $|\alpha| R_x$.
  - If x(n) is multiplied by a complex exponential, $e^{jn\omega_0}$,

$$e^{jn\omega_0} x(n) \overset{Z}{\leftrightarrow} X\left(e^{-j\omega_0} z\right)$$

Properties of Z-Transform (cont.)

- **<u>Convolution Theorem:</u>**
  - Convolution in the time-domain is mapped into multiplication in frequency domain.

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z) X_2(z)$$

  - $Y(z)=X_1(z)\ X_2(z)$
  - The ROC includes the intersection of $X_1(z)$ and $X_2(z)$.
  - If a pole that borders on the ROC of one of the *z*-transforms is canceled by a zero of the other, then the region of convergence of Y(z) may be larger.

Example #1

- Consider the two sequences x(n)= $\alpha^n u(n)$ and h(n)= $\delta(n) - \alpha\delta(n-1)$.
  - The z-transform of x(n) is:

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

  - And the z-transform of h(n) is:

$$H(z) = 1 - \alpha z^{-1}, \quad 0 < |z|$$

  - However, the z-transform of the convolution of x(n) with h(n) is :

$$Y(z) = X(z)H(z) = \frac{1}{1-\alpha z^{-1}}.\left(1-\alpha z^{-1}\right) = 1$$

  - Which is due to pole-zero cancellation, has a ROC that is the entire z-plane.

Properties of Z-Transform (cont.)

- **Conjugation:**
  - If X(z) is the *z*-transform of x(n), the *z*-transform of the complex conjugate of x(n) is:

$$x^*(n) \overset{z}{\longleftrightarrow} X^*(z^*)$$

  - If x(n) is real valued x(n)=x*(n), then X(z)=X*(z*).

- **Derivative:**
  - If X(z) is the *z*-transform of x(n), the *z*-transform of the nx(n) is:

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

  - Repeated application of this property allows for the evaluation of the *z*-transform of $n^k x(n)$ for any integer k.

Table of Properties of Z-Transform

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$

Table of Properties of Z-Transform

$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Initial-value theorem:

$x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$

Example #2

- Find the *z*-transform of $x(n)=n\alpha^{n}u(-n)$:
- Solution:

$$X(z) = -\frac{\alpha^{-1}z}{\left(1-\alpha^{-1}z\right)^{2}}, \quad |z| < \alpha$$

Initial-Value Theorem

- If x[n] is zero for n<0 (i.e., if x[n] is causal), then:

$$x[0] = \lim_{z \to \infty} X(z)$$

- If we let z➔∞, each term in X(z) goes to zero except the first.

Inverse Z-Transform

- Z-transform is a useful tool in linear system analysis.
- As it is important to find *z*-transform of a sequence, there are methods used to invert the *z*-transform and recover the sequence x(n) form X(z).
- Some of procedure for inverse *z*-transform are as follows:
  - Inspection Method
  - Partial Fraction Expansion
  - Power Series Expansion

Inspection Method

- It consists simply of becoming familiar with or recognizing “by inspection”, certain transform pairs.
- For example,

$$a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

- If we need to find the inverse *z*-transform of:

$$X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}} \right), \quad |z| > \frac{1}{2}$$

Partial Fraction Expansion

- Assume that X(z) is expressed as a ratio of polynomials in z^{-1}, i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \rightarrow eq(1)$$

- An equivalent expression is:

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} \rightarrow eq(2)$$

Partial Fraction Expansion (cont.)

- Eq.(2) shows that there will be M zeros and N poles at nonzero locations in the *z*-plane.
- There will be either M-N poles at z=0 if M>N or N-M zeros at z=0 if N>M.
- In other words eq.(1) will always have the same number of poles and zeros in the finite *z*-plane and there are no poles or zeros at z=∞.
- To obtain the partial fraction expression of X(z) in eq.(1), the X(z) could be expressed in the form:

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)} \rightarrow eq(3)$$

Partial Fraction Expansion (cont.)

- Where c_k's are the nonzero zeros of X(z) and the d_k's are the nonzero poles of X(z).

- If M<N and the poles are all first order, then X(z) can be expressed as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \rightarrow eq(4)$$

- Coefficient A_k can be found from:

$$A_k = \left(1 - d_k z^{-1}\right) X(z)\Big|_{z=d_k} \rightarrow eq(5)$$

Example #3

- Consider a sequence x[n] with *z*-transform:

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \quad |z| > \frac{1}{2}$$

- Solution:

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Example #4

- **Inverse by Partial Fractions:**
  - Consider a sequence x[n] with *z*-transform:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}, \quad |z| > 1$$

  - Solution:

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power Series Expansion

- The *z*-transform is a power series expansion,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \ldots. + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots.$$

- Where the sequence values of x(n) are the coefficients of z^{-n} in the expansion.

Example #5

- **Finite-Length Sequence:**
  - Suppose X(z) is given in the form:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

  - Solution:

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Example #6

- **Power Series Expansion by Long Division:**
  - Consider the *z*-transform:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

  - Solution:

$$x[n] = a^n u[n]$$

Example #7

- The *z*-transform of a sequence x(n) is: (4.1)

$$X(z) = \frac{z + 2z^{-2} + z^{-3}}{1 - 3z^{-4} + z^{-5}}$$

- If the ROC includes the unit circle, find the DTFT of x(n) at $\omega = \pi$.

- Solution:

$$X\left(e^{j\omega}\right)\Big|_{\omega=\pi} = 0$$

Example #8

- Find the *z*-transform of the following sequences: (4.2, 4.4)

$$(a) \quad x[n] = \cos(n\omega_0)u(n)$$

$$(b) \quad x[n] = \left(\frac{1}{3}\right)^n u(-n)$$

$$(c) \quad x[n] = \left(\frac{1}{2}\right)^n u(n+2) + (3)^n u(-n-1)$$

Example #8 (cont.)

- Solution:

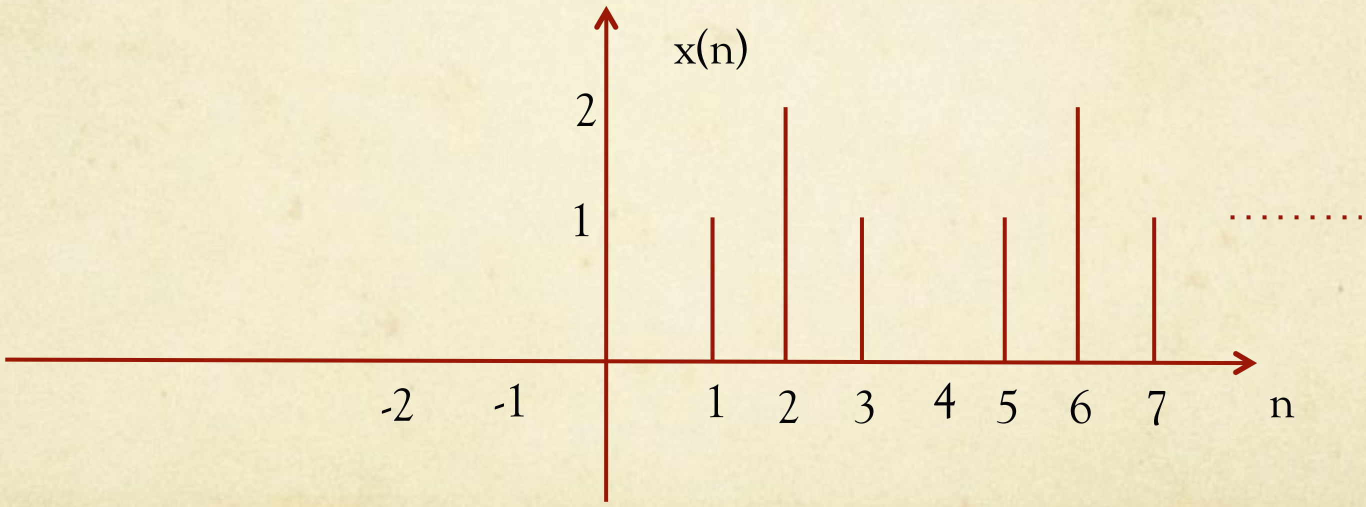
$$(a)\quad X(z)=\frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}},\quad |z|>1$$

$$(b)\quad Y(z)=\frac{1}{1-3z^{-1}},\quad |z|>3$$

$$(c)\quad X(z)=\frac{4z^2}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-3z^{-1}},\quad \frac{1}{2}<|z|<3$$

Example #9

- Consider the sequence shown in the figure below. The sequence repeats periodically with a period N=4 for n≥0 and is zero for n<0. find the *z*-transform of this sequence along with its ROC. (4.9)



Example #10

- The deterministic autocorrelation sequence corresponding to a sequence x(n) is defined as: (4.39)

$$r_x(n) = \sum_{k=-\infty}^{\infty} x(k)x(n+k)$$

- (a) Express $r_x(n)$ as the convolution of two sequences, and find the *z*-transform of $r_x(n)$ in terms of the *z*-transform of x(n).

- (b) If $x(n)=\alpha^n u(n)$, where $|a|<1$, find the autocorrelation sequence, $r_x(n)$ and its *z*-transform.

Example #10 (cont.)

- Solution:

$$(a) \quad R_x(z) = X(z)X\left(z^{-1}\right)$$

$$(b) \quad r_x(n) = \frac{1}{1-a^2}\left[a^n u(n) + a^{-n}u(-n-1)\right] = \frac{1}{1-a^2}a^{|n|}, \quad |a| < z < 1/|a|$$

Example #11

- Find the one-sided *z*-transform of the following sequence: (4.30)

$$x(n) = \left(\frac{1}{3}\right)^3 u(n+3)$$

- Solution:

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$