Digital Signal Processing
Sampling-I

Lecture-7
06-April-16

Introduction

- Most Discrete time signals come from sampling a continuous time signal, such as speech and audio signals, radar and sonar data, seismic and biological signals.
- The process of converting these signals into digital form is called analog to digital (A/D) conversion.
- The process of reconstructing an analog signal from its samples is known as digital to analog (D/A) conversion.

Periodic Sampling

- The typical method of obtaining a discrete time representation of a continuous time signal is through periodic sampling.

- Sequence of samples x[n] is obtained from a continuous time signal $x_c(t)$ according to the relation:

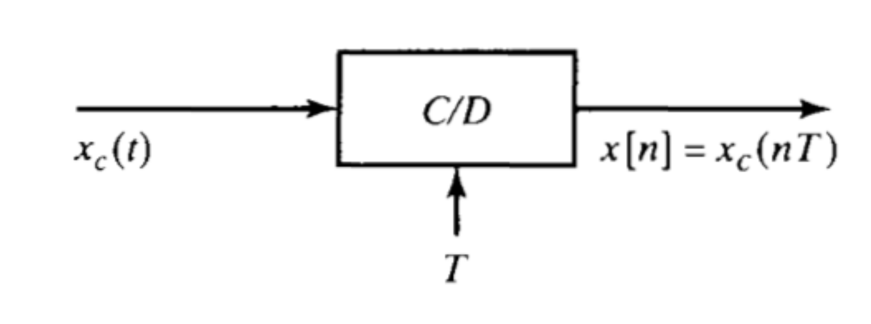
$$x[n] = x_c(nT), \quad -\infty < n < \infty \quad \rightarrow eq(1)$$

- In eq.(1), T is the sampling period and its reciprocal $f_s=1/T$ is the sampling frequency in samples per second.

- When frequency in radians per second is used it is expressed as $\Omega_s=2\pi/T$.

Periodic Sampling(cont.)

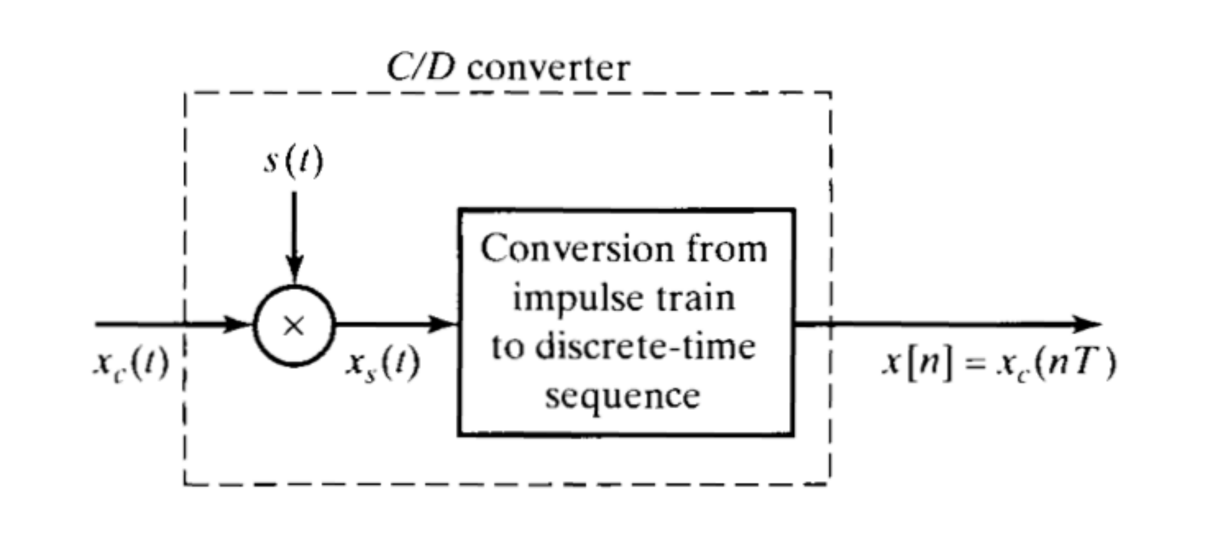
- A system that implements the operation of eq.(1) as an ideal continuous-to-discrete-time (C/D) converter is shown below:



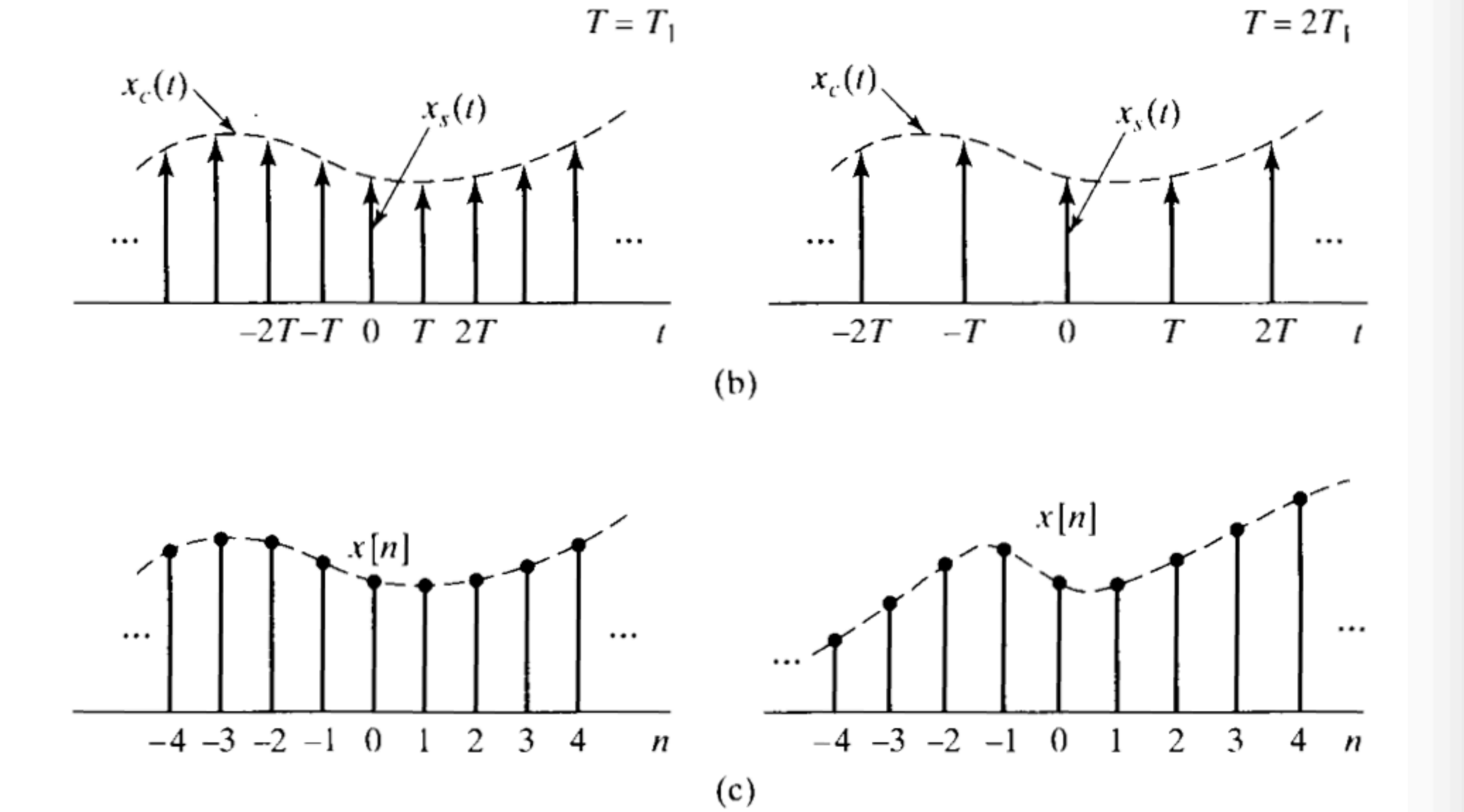
- The sampling operation is generally not invertible i.e., given the output x[n], it is not possible in general to reconstruct $x_c(t)$.

Periodic Sampling(cont.)

- It is convenient to represent the sampling process mathematically in two stages.
- The stages consist of an impulse train modulator followed by conversion of the impulse train to a sequence.



Periodic Sampling(cont.)



Frequency Domain Representation of Sampling

- To derive the frequency domain relation between the input and output of an ideal C/D converter.
- Let us first consider the conversion of $x_c(t)$ to $x_s(t)$ through modulation of the periodic impulse train:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Where $\delta(t)$ is the unit impulse function or Dirac delta function. We modulate s(t) with $x_c(t)$,

$$x_s(t) = x_c(t)s(t)$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Representation of Sampling (cont.)

- Through the sifting property of the impulse function, $x_s(t)$ can be expressed as:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

- Let us now consider the Fourier transform of $x_s(t)$.
- $x_s(t)$ is the product of $x_c(t)$ and $s(t)$, the Fourier transform of $x_s(t)$ is the convolution of the Fourier transforms $X_c(j\Omega)$ and $S(j\Omega)$.
- The Fourier transform of a periodic impulse train is a periodic impulse train.
- Specifically,

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

Frequency Domain Representation of Sampling (cont.)

- Where $\Omega_s = 2\pi/T$ is the sampling frequency in radians/s. Since,

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

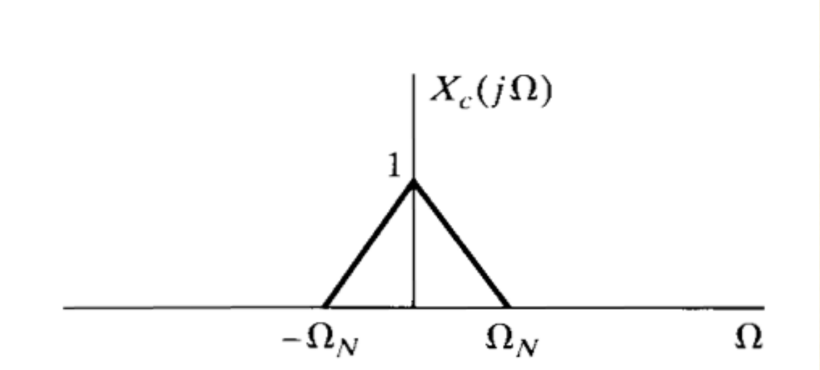
- Where * denotes the operation of continuous-variable convolution. It follows that:

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - k\Omega_s)\right)$$

- Above equation provides the relationship between the Fourier transforms of the input and the output of the impulse train modulator.

Frequency Domain Representation of Sampling (cont.)

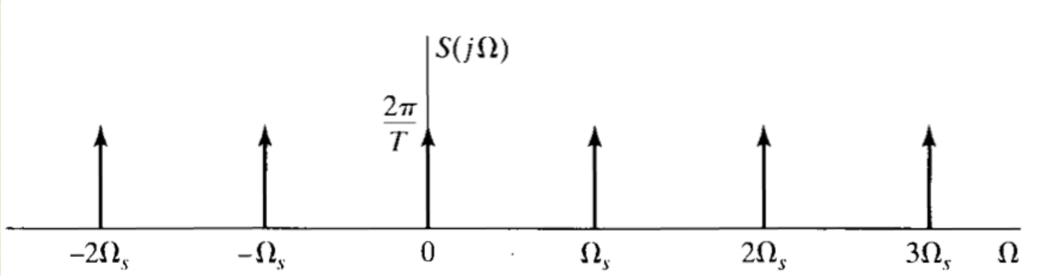
- The figures below depicts the frequency domain representation of impulse train of samples.



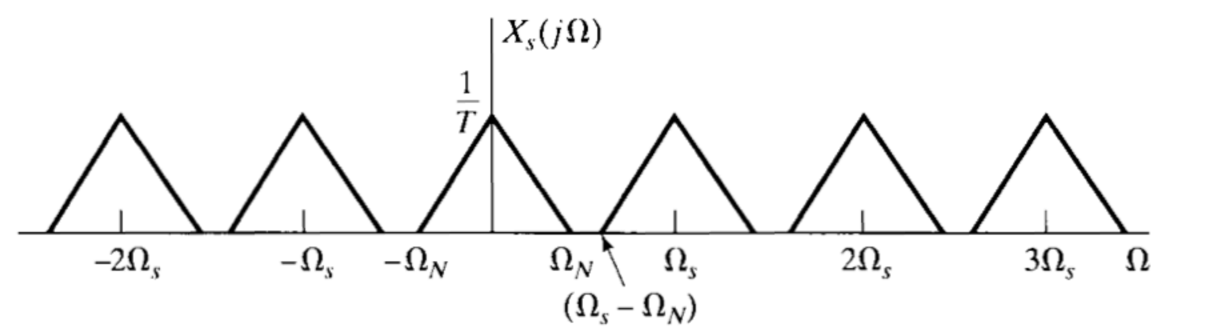
- The above figure represents a band limited Fourier transform whose highest nonzero frequency component in $X_c(j\Omega)$.

Frequency Domain Representation of Sampling (cont.)

- Figure below shows the periodic impulse train $S(j\Omega)$.

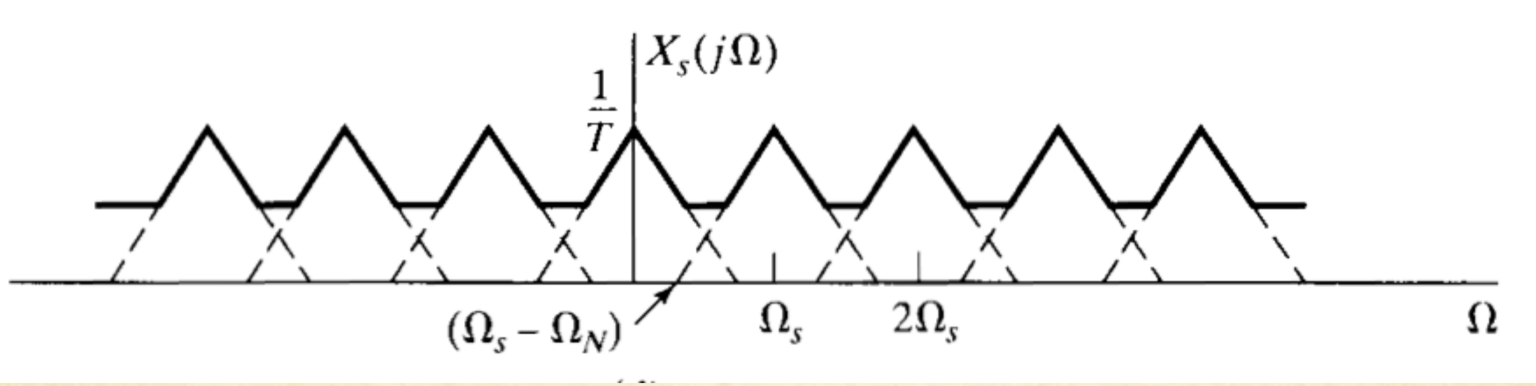


- $X_s(j\Omega)$, the result of convolving $X_c(j\Omega)$ with $S(j\Omega)$.



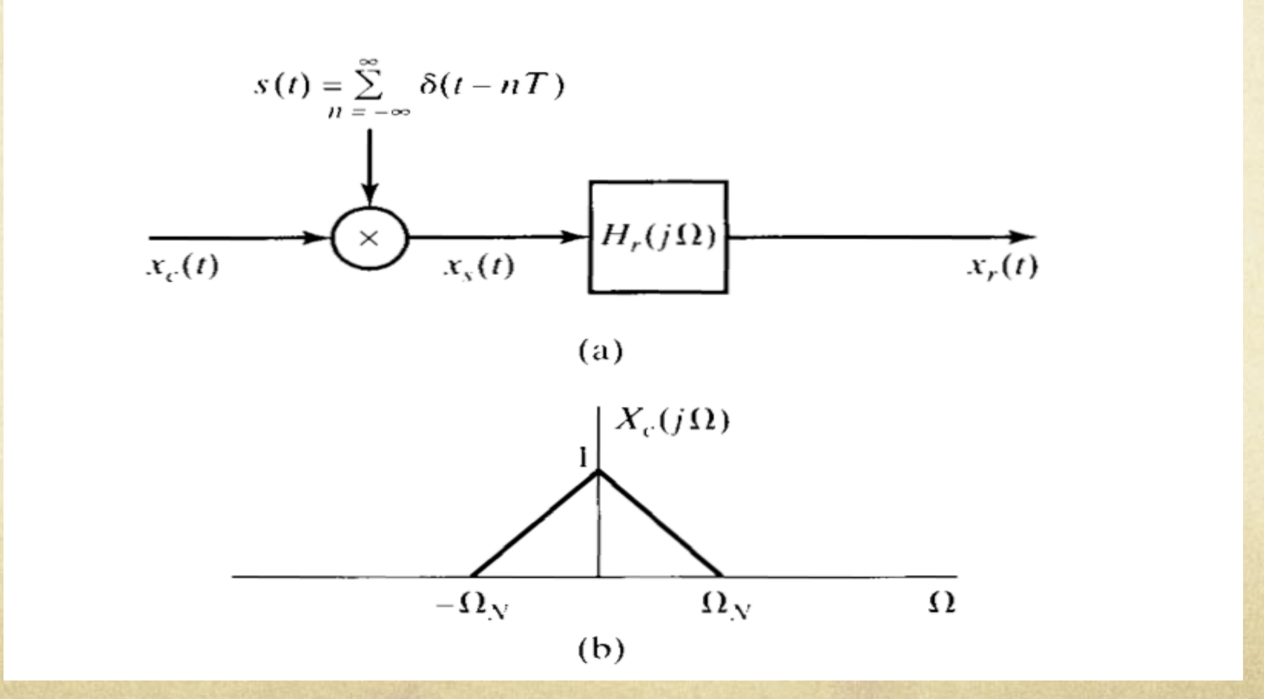
$$\Omega_s - \Omega_N > \Omega_N \quad or \quad \Omega_s > 2\Omega_N$$

Frequency Domain Representation of Sampling (cont.)

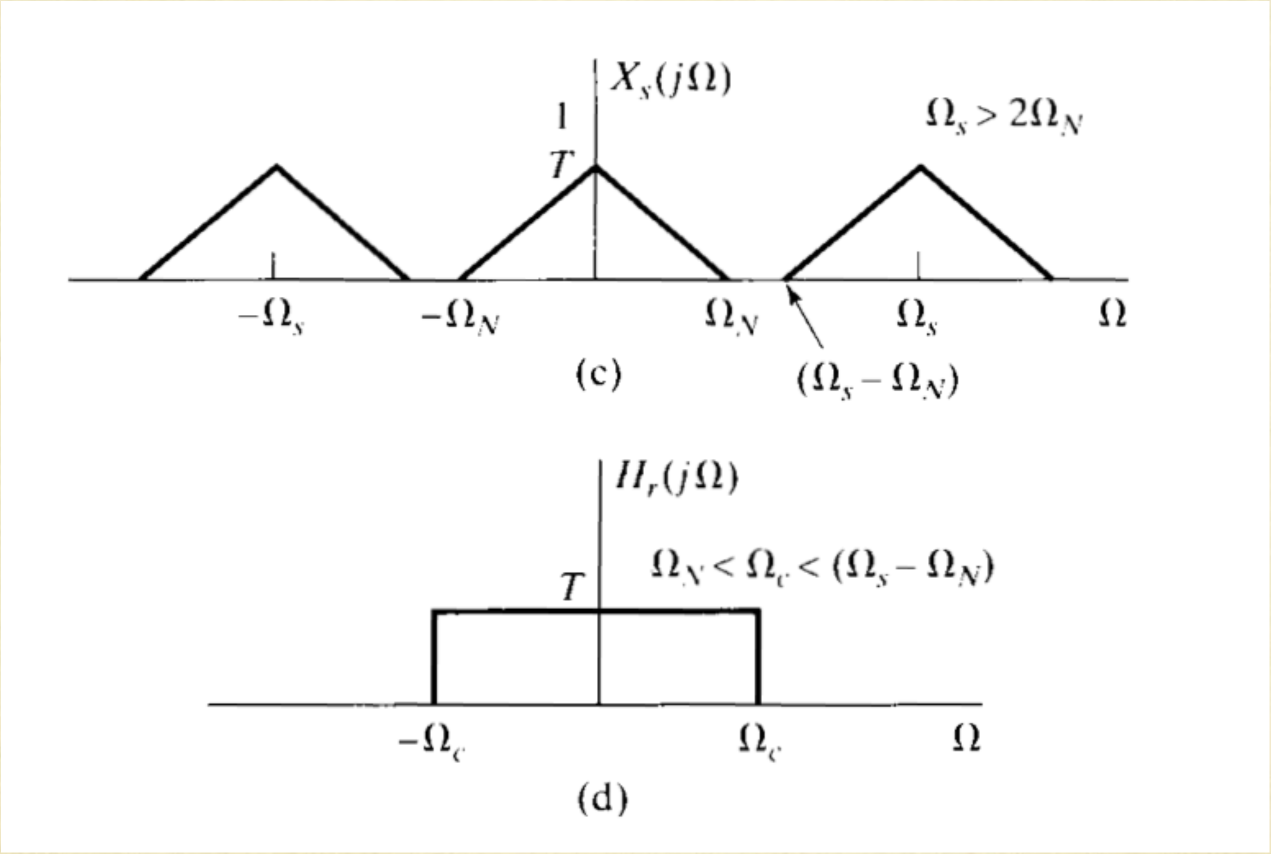


Frequency Domain Representation of Sampling (cont.)

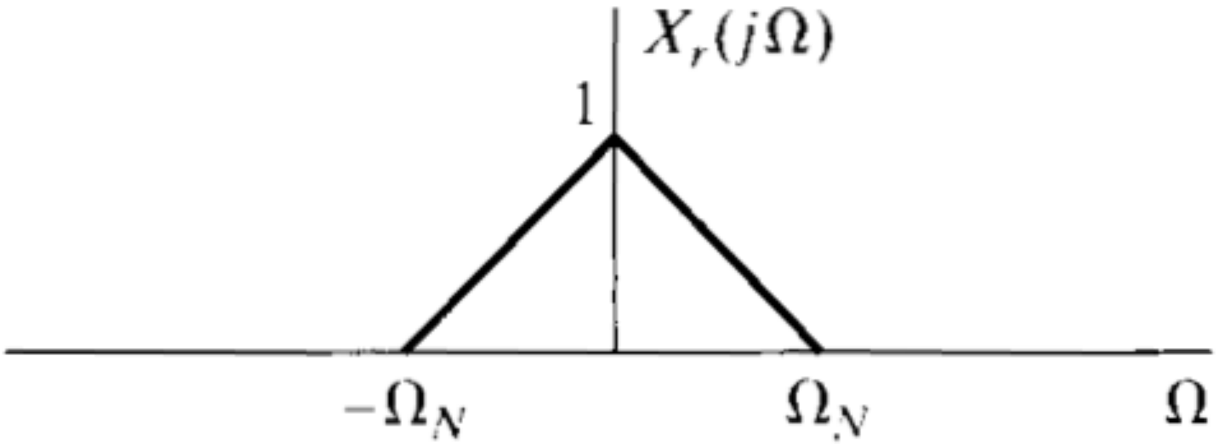
- $x_c(t)$ can be recovered from $x_s(t)$ with an ideal low pass filter.
- This is depicted in following figures.



Frequency Domain Representation of Sampling (cont.)



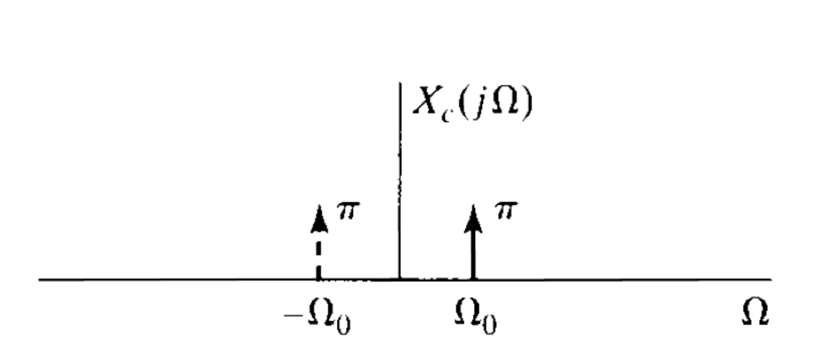
Frequency Domain Representation of Sampling (cont.)



(e)

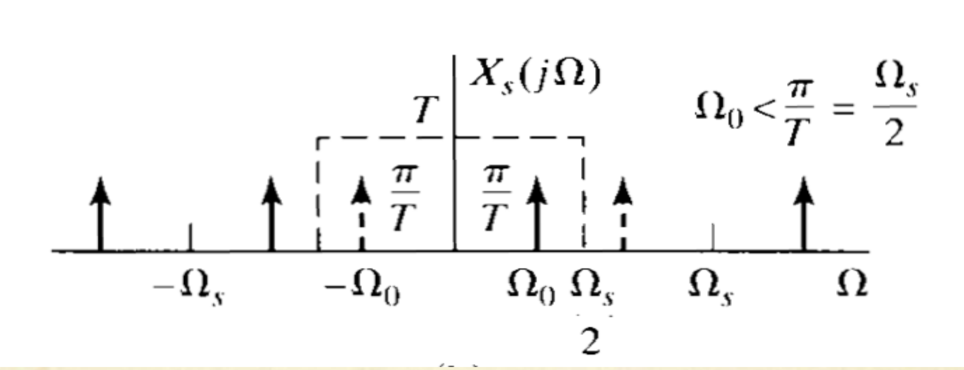
Aliasing

- Aliasing in the frequency domain is depicted in the following example for the simple case of a cosine signal.
- The figure below shows the Fourier transform of the signal : $x_c(t)=\cos \Omega_0 t$.

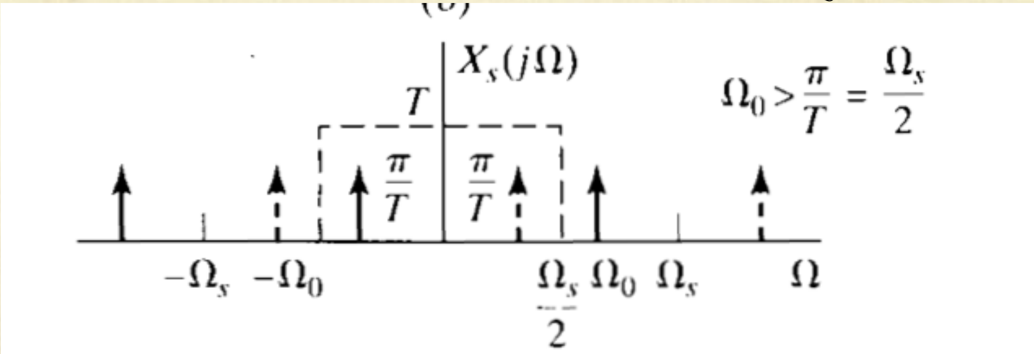


Aliasing

- The following figure shows the Fourier transform of $x_s(t)$ with $\Omega_0 < \Omega_s/2$.

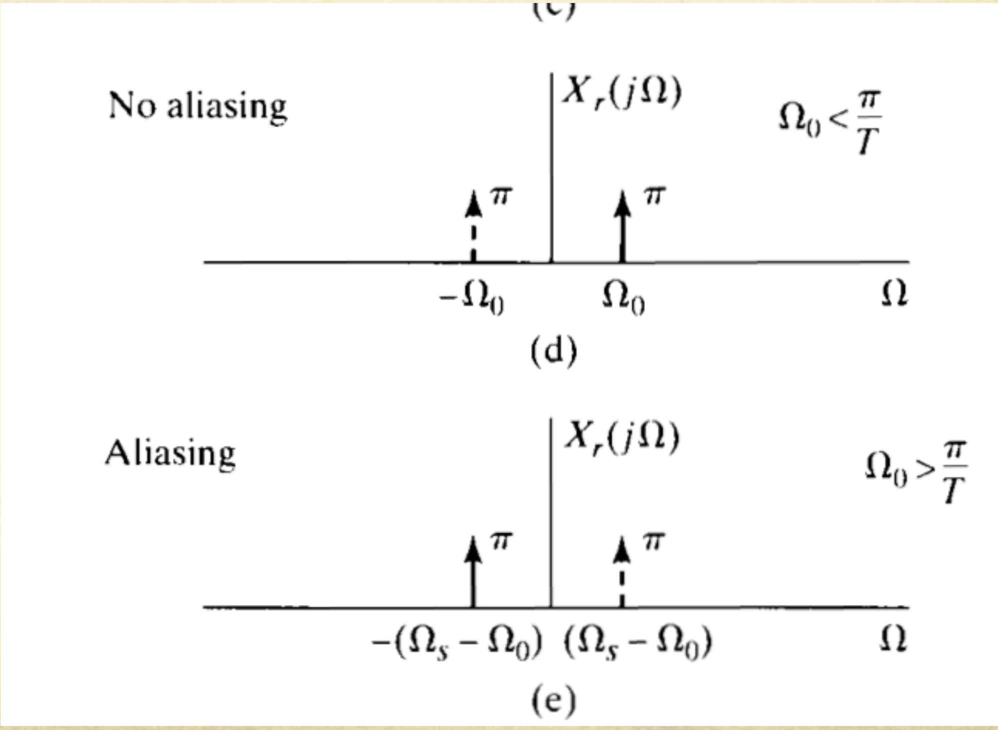


- Figure below shows the Fourier transform of $x_s(t)$ with $\Omega_0 > \Omega_s/2$.



Aliasing

- The Fourier transform of the low pass filter output for $\Omega_0 < \Omega_s/2 = \pi/T$ and $\Omega_0 > \pi/T$, respectively with $\Omega_c = \Omega_s/2$, shown below:



Aliasing

- Above two figures correspond to the case of aliasing.
- With no aliasing the reconstructed output is: $x_r(t)=\cos \Omega_0 t$.
- With aliasing the reconstructed output is: $x_r(t)=\cos (\Omega_s-\Omega_0)t$.
- That is the higher the frequency signal $\cos \Omega_0 t$ has taken on the identity (alias) of the lower frequency signal $\cos (\Omega_s-\Omega_0)t$ as a consequence of the sampling and reconstruction.
- This above discussion if the basis for the Nyquist sampling theorem.

Nyquist Sampling Theorem

- Let $x_c(t)$ be a band limited signal with:

$$X_c(j\Omega) = 0 \quad for \quad |\Omega| \geq \Omega_N$$

- Then $x_c(t)$ is uniquely determined by its samples $x[n]=x_c(nT)$, $n=0,\pm1,\pm2,\ldots$. If: $\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$
- The frequency Ω_N is commonly referred to as the Nyquist frequency and the frequency $2\Omega_N$ that must be exceeded by the sampling frequency is called the Nyquist rate.

Example #1

- Sampling and reconstruction of a sinusoidal signal.
  - If we sample the continuous time signal $x_c(t)=\cos(4000\pi t)$ with sampling period T=1/6000.

$$x[n] = x_c(nT) = \cos(4000\pi Tn) = \cos(\omega_0 n)$$

  - Where,

$$\omega_0 = 4000\pi T = \frac{2\pi}{3}$$

  - In this case,

$$\Omega_s = \frac{2\pi}{T} = 12000\pi$$

  - The highest frequency of the signal is: $\Omega_0 = 4000\pi$
  - So the conditions of the Nyquist sampling theorem are satisfied and there is no aliasing.
  - The Fourier transform of $x_c(t)$ is:

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

Example #1 (cont.)