Digital Signal Processing
Transform Analysis of LTI Systems

Lecture-10
19-April-16

Transform Analysis of LTI Systems

- For LTI systems we can write:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[n] * h[n]$$

- In *z*-transform the input output relation can be written as:

$$Y(z) = H(z)X(z)$$

- Where H(z) is the system function.
- LTI system is completely characterized by its system function or equally its impulse response.

Frequency Response of LTI Systems

- The frequency response $H(e^{j\omega})$ of an LTI system is defined as the complex gain (Eigen value) that the system applies to the complex exponential input (Eigen function) $e^{j\omega n}$.

- The Fourier transform of the system input and output are therefore related by: $Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$

- In terms of magnitude and phase,

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

- Where $|H(e^{j\omega})|$ is magnitude response or gain of the system and $<H(e^{j\omega})$ is the phase response or phase shift.

Ideal Low Pass Filter

- Frequency components of the input are suppressed in the output if $|H(e^{j\omega})|$ is small at those frequencies.
- The ideal low pass filter is defined as the LTI system with frequency response:

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

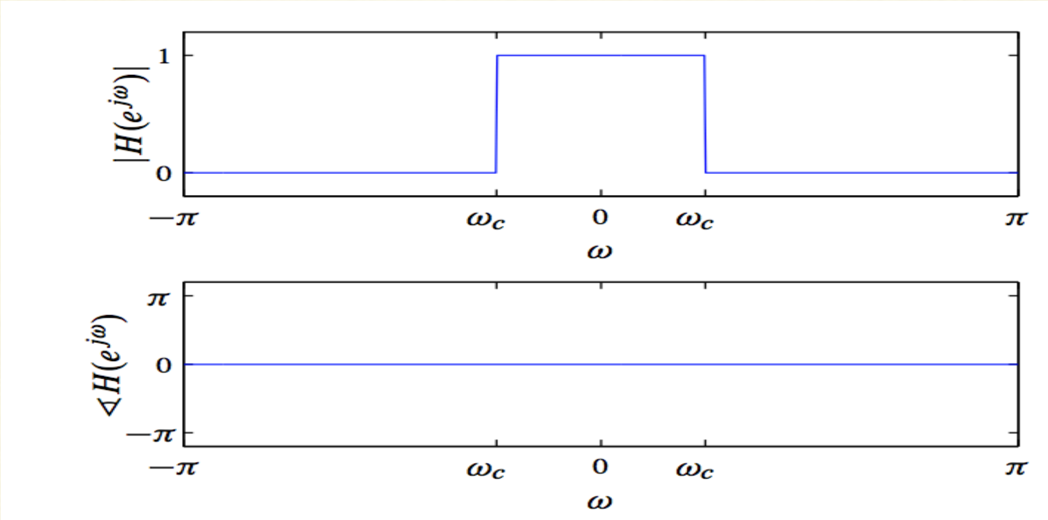
- $H_{lp}(e^{j\omega})$ is also periodic with period 2π.
- Its impulse response for $-\infty<n<\infty$ is:

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right) = \frac{\sin(\omega_c n)}{\pi n}$$

Ideal Low Pass Filter (cont.)

- Its magnitude and phase are:



- Ideal low pass filter is non-causal.
- The phase response of ideal low pass filter is specified to be zero.
- If it were not zero, the low-frequency band selected by the filter would also have phase distortion.

Ideal High-Pass Filter

- The ideal high-pass filter is defined as:

$$H_{hp}\left(e^{j\omega}\right) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c < |\omega| \le \pi \end{cases}$$

- Since, $H_{hp}\left(e^{j\omega}\right) = 1 - H_{lp}\left(e^{j\omega}\right)$ its frequency response is:

$$h_{hp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}$$

- The ideal high-pass filter passes the frequency band $\omega_c < \omega \le \pi$ undistorted and rejects frequency below ω_c.

Phase Distortion & Delay

- Consider the ideal delay with impulse response

$$h_{id}[n] = \delta[n - n_d]$$

- And the frequency response is:

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

- Or,

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega n_d \qquad |\omega| < \pi$$

- The phase distortion of the ideal delay is therefore a linear function of ω.

- In many applications delay distortion would be considered a rather mild form of phase distortion, since its effect is only to shift the sequence in time.

Phase Distortion & Delay (cont.)

- A filter with linear phase response can be viewed as a cascade of a zero-phase filter, followed by a time shift or delay.
- The ideal low-pass filter with phase distortion would be defined as:

$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega n_d} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

- With impulse response:

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}$$

- Filters high-frequency components and delays signal by n_d.
- Linear-phase ideal low-pass filters is still not implementable.

Phase Distortion & Delay (cont.)

- A convenient measure of the linearity of the phase is the group delay.
- The basic concept of group delay relates to the effect of the phase on a narrowband signal.

System Response for LCCD Systems

- Ideal filters cannot be implemented with finite computation.
- Therefore we need approximations to ideal filters.
- Constant-coefficient difference equations are:
  - General to represent most useful systems.
  - Implementable.
  - LTI and causal with zero initial conditions.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The *z*-transform is useful in analyzing difference equations.

System Response for LCCD Systems

- Taking *z*-transform of both sides:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

- The system function for a system that satisfies a difference equation of the required form is therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} \left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - d_k z^{-1}\right)}$$

Example #1

- Given the system function:

$$H(z) = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}$$

- The corresponding difference equation can be expressed as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}}$$

- Therefore,

$$\left(1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)Y(z) = \left(1+2z^{-1}+z^{-2}\right)X(z)$$

- And the difference equation is:

$$y[n]+\frac{1}{4}y[n-1]+\frac{3}{8}y[n-2] = x[n]+2x[n-1]+x[n-2]$$

Stability & Causality

- A difference equation does not uniquely specify the impulse response of a LTI system.
  - Need to know the ROC.
- Properties of system gives clues about the ROC.
- Causal systems must be right sided.
  - ROC is outside the outermost pole.
- Stable system requires absolute summable impulse response:

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

  - Absolute summability implies existence of DTFT.
  - DTFT exists if unit circle is in the ROC.
  - Therefore stability implies that the ROC include the unit circle.

Stability & Causality

- Causal and stable systems have all poles inside unit circle.
  - Causal hence the ROC is outside outermost pole.
  - Stable hence unit circle included in ROC.
  - This means outermost pole is inside unit circle.
  - Hence all poles are inside unit circle.

Example #2

- Consider the LTI system with input and output related through the difference equation:

$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$

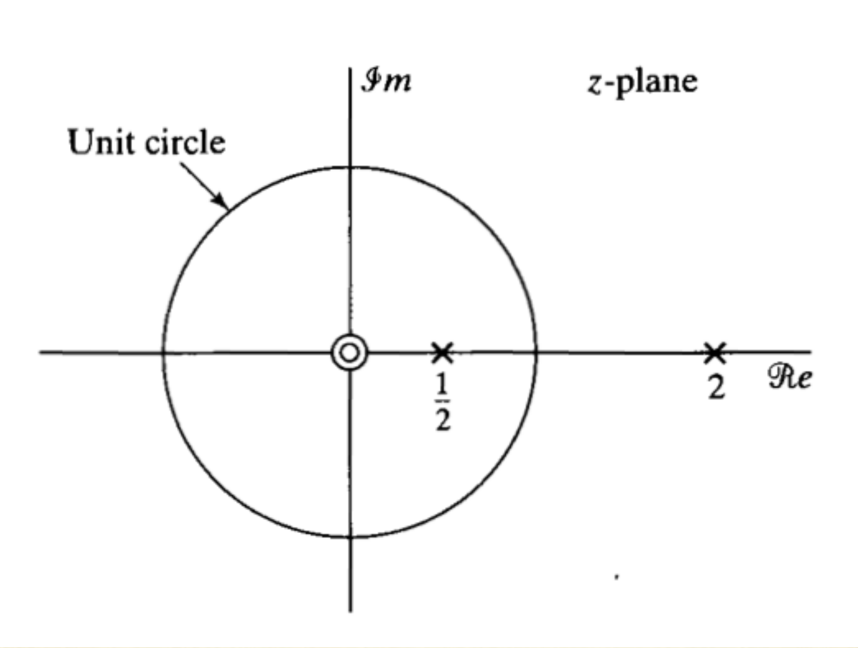
- Frequency response H(z) is given by:

$$H(z) = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

- There are 3 choices of ROC:
  - Causal: ROC outside of outermost pole $|z|>2$ (but then not stable).
  - Stable: ROC such that $1\backslash 2<|z|<2$. (but then not causal).
  - If $|z|<1/2$ then the system is neither causal nor stable.

Example #2 (cont.)

- For a causal and stable system the ROC must be outside the outermost pole and include the unit circle. This is only possible if all the poles are inside the unit circle.



Inverse System

- The system $H_i(z)$ is the inverse system to $H(z)$ if:

$$G(z) = H(z)H_i(z) = 1$$

- Which implies that:

$$H(z) = \frac{1}{H_i(z)}$$

- The time-domain equivalent is:

$$g[n] = h[n] * h_i[n] = \delta[n]$$

- If it exists the frequency response of the inverse system is:

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

Inverse System

- Not all systems have an inverse: zeros cannot be inverted.
- The ROC of $H(z)$ and $H_i(z)$ must overlap.

Example #3

- Inverse system for first order system:
- Let H(z) be:

$$H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}}$$

- With ROC $|z|>0.9$. Then $H_i(z)$ is:

$$H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

- Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC, and the only choice for the ROC of $H_i(z)$ that overlaps with $|z|>0.9$ is $|z|>0.5$.
- Therefore the impulse response of the inverse system is:

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

- In this case the inverse system is both causal and stable.

Minimum Phase

- A LTI system is stable and causal with a stable and causal inverse if and only if both the poles and zeros of $H(z)$ are inside the unit circle.
- Such systems are called minimum phase systems.

Impulse Response for Rational System Functions

- If a system has a rational transfer function, with at least one non-zero pole of H(z) that is not cancelled by a zero, then there will always be a term corresponding to an infinite length sequence in the impulse response.

- Such a systems are called infinite impulse response (IIR) systems.

- If a system has no poles except at z=0, then:

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- In this case the system is determined to within a constant multiplier by its zeros, so the impulse response has a finite length:

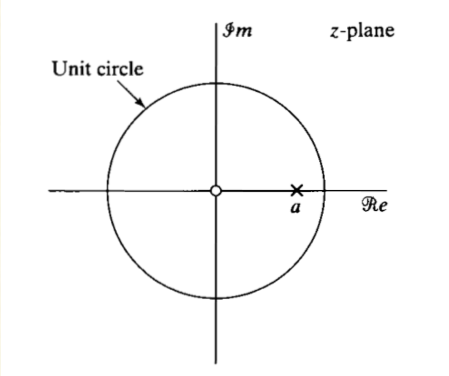
$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

- The impulse response is finite in length and the system is called a finite impulse response (FIR) system.

Example #4

- A first order IIR system:
- Given a causal system satisfying the difference equation:

$$y[n]-ay[n-1]=x[n]$$



- The system function is: (by inspection)

$$H(z)=\frac{1}{1-az^{-1}}=\frac{z}{z-a},\quad |z|>|a|$$

- The condition for stability is |a|<1. The inverse *z*-transform is:

$$h[n]=a^n u[n]$$

Example #5

- A simple FIR system:
- Consider the truncated impulse response:

$$h[n] = \begin{cases} a^n & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

- The system function is:

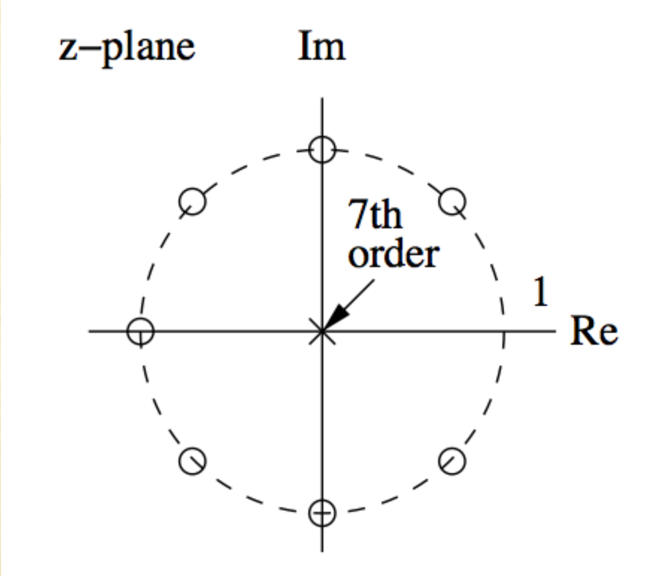
$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}$$

- The zeros of the numerator are at:

$$z_k = ae^{j2\pi k/(M+1)}, \quad k = 0,1,.....,M$$

Example #5 (cont.)

- With a assumed real and positive, the pole at z=a is cancelled by a zero.
- The pole-zero plot for the case of M=7 is therefore given by:



Example #5 (cont.)

- The difference equation satisfied by the input and output of the LTI system is the convolution:

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]$$

- The input and output also satisfy the difference equation:

$$y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$$