-Digital Signal Processing-
Digital Filter Design-I

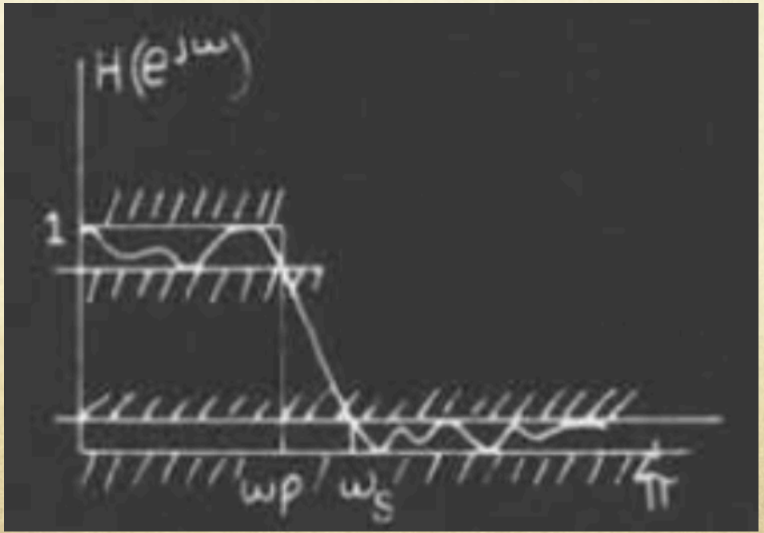
Lecture-14
11-May-16

Digital Filter Design

- Complex exponential are Eigen functions or in other words with complex exponential input the output is a complex exponential of the same complex frequency multiplied by the system function $H(e^{j\omega})$.

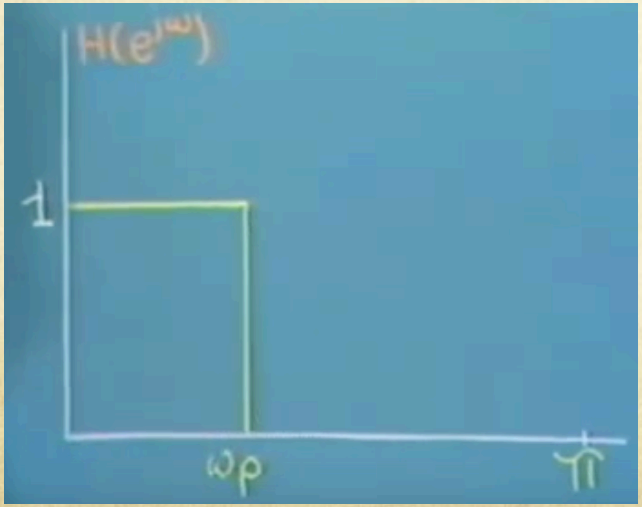
$$e^{j\omega_0 n} \rightarrow H\left(e^{j\omega_0}\right)e^{j\omega_0 n}$$

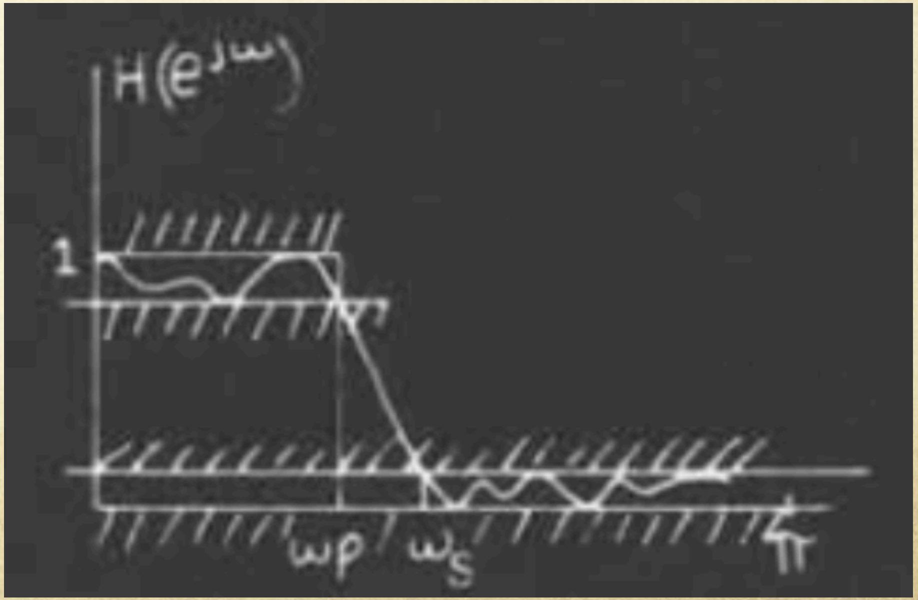
$$\cos\omega_0 n \rightarrow |H|\cos(\omega_0 n + \theta)$$



Digital Filter Design (cont.)

- For example, if we had a signal which is sum of two components, one of which is of low frequency and we have to keep that one.
- The other one which one is of high frequency and we want to suppress that one.
- To extract the signal we want, we will consider a system, whose frequency response ideally is unity at the low frequency end and zero at the high frequency end or after some cut of frequency ω_p.
- Generally referred to as selective filter.





Digital Filter Design (cont.)

- Digital filter design problem is: to design a rational transfer function which approximate in some sense the ideal filter maintaining the specifications of pass band and stop band deviation.

- There are several classes of design techniques that we consider and which are:
  - Analytical:
    - It permits the approximation of the ideal frequency characteristics. The approximation is carried out through analytical procedure and results in a closed form transfer function.
  - Continuous time ➔ Discrete time (Mapping):
    - Reason for mapping is that a lot of work have developed for the design of continuous time filters. If it is possible to map those designs than we should simply take the advantage of the results that were developed in the design of analog filters.
  - Algorithmic (computer aided):
    - There are a variety of algorithmic procedure that we can go through to carry out the digital filter design.

Design Techniques

- First we will discuss about IIR filter designs or also known as recursive digital filters.
- Continuous ➔ Discrete

$$H_a(s) \rightarrow H(z)$$

$$h_a(t) \rightarrow h(n)$$

  - (a): jΩ-axis (s-plane) ➔ unit circle (z-plane)
  - (b): $H_a(s)$ {Stable} ➔ H(z) {Stable}

Differential→ Differences

- It is a method which corresponds to going from analog filters to digital filters by mapping differentials equations in analog domain to differences in digital domain.

$$H_a(s)$$

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{dt^k}$$

$$y_a(t) \rightarrow y(n)$$

$$\left.\frac{dy_a(t)}{dt}\right|_{t=nT} \rightarrow \Delta^{(1)}\left[y(n)\right]$$

$$\Delta^{(1)}\left[y(n)\right] = \frac{y(n+1) - y(n)}{T}$$

Differential➔ Differences (cont.)

$$\frac{d^k y_a(t)}{dt^k} \rightarrow \Delta^{(k)}\left[y(n)\right]$$

$$\Delta^{(k)}\left[y(n)\right] = \Delta^{(1)}\left[\Delta^{(k-1)} y(n)\right]$$

$$\sum_{k=0}^{N} c_k \Delta^{(k)}\left[y(n)\right] = \sum_{k=0}^{M} d_k \Delta^{(k)}\left[x(n)\right]$$

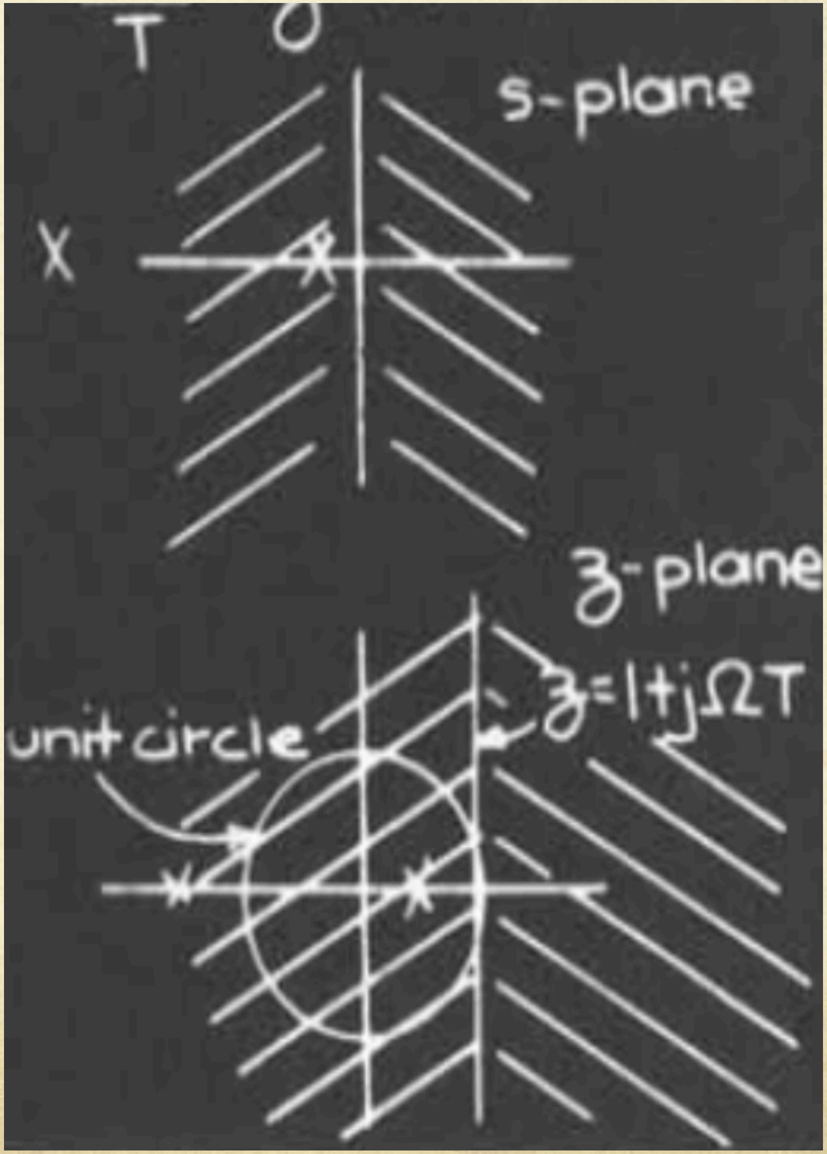
$$L\left[\frac{dy_a(t)}{dt}\right] = sY_a(s)$$

$$Z\left[\frac{y(n+1) - y(n)}{T}\right] = \frac{z-1}{T} Y(z)$$

Differential➔ Differences (cont.)

$$H(z) = H_a(s)\Big|_{s=\frac{z-1}{T}}$$

$$s = \frac{z-1}{T} \Rightarrow z = 1 + sT$$



Impulse Invariance

- It's a method which converts an analog filter to a digital filter simply by choosing the unit sample response of the digital filter to be equally spaced samples of the impulse response of the analog filter.

$$h(n) = h_a(nT)$$

$$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left[\frac{j\omega}{T} + \frac{j2\pi k}{T}\right]$$

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} u(n)$$

Impulsive Invariance (cont.)

$$h(n) = \sum_{k=1}^{N} A_k \left(e^{s_k T}\right)^n u(n)$$

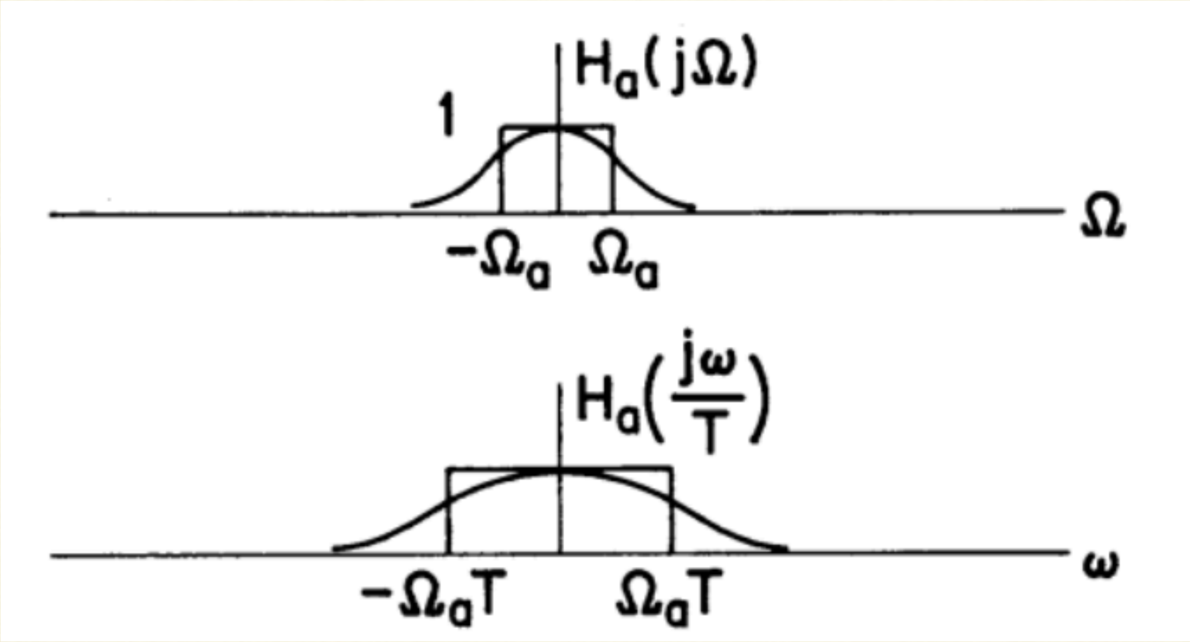
$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

$$\begin{matrix} pole \quad at \\ s = s_k \end{matrix} \quad \Rightarrow \quad \begin{matrix} pole \quad at \\ z = e^{s_k T} \end{matrix}$$

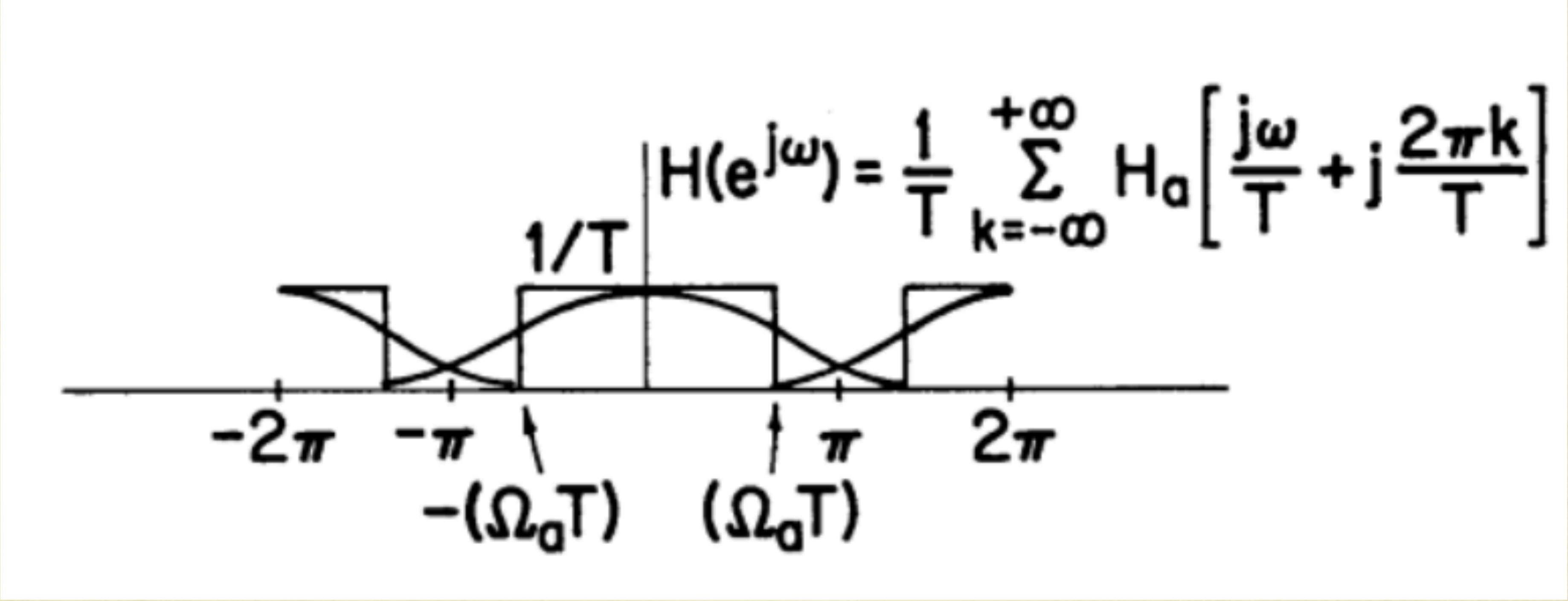
$$\left.\begin{matrix} s_k = \sigma_k + j\Omega_k \\ \left|z_k\right| = \left|e^{\sigma_k T}\right|\left|e^{j\sigma_k T}\right| \end{matrix}\right\} R_e\left[s_k\right] < 0 \Rightarrow \left|z_k\right| < 1$$

Impulsive Invariance (cont.)

- An analog frequency response and the corresponding digital frequency response obtained through impulse invariance:



Impulsive Invariance (cont.)



Example #1

- Consider an analog filter for which the input $x_a(t)$ and output $y_a(t)$ are related by the linear constant-coefficient differential equation:

$$\frac{dy_a(t)}{dt} + 0.9 y_a(t) = x_a(t)$$

- A digital filter is obtained by replacing the first derivative by the first forward difference so that with x(n) and y(n) denoting the input and output of the digital filter:

$$\left[\frac{y(n+1) - y(n)}{T}\right] + 0.9 y(n) = x(n)$$

- Throughout this problem the digital filter is assumed to be causal.
  - Determine and sketch the magnitude of the frequency response of the analog filter.
  - Determine and sketch the magnitude of the frequency response of the digital filter for T=10/9.
  - Determine the range of values of T for which the digital filter is unstable. (Note that the analog filter is stable).

Example #1 Solution

- (a): Applying the Laplace transform to both sides of the differential equation we obtain:

$$Y_a(s)\left[s+0.9\right]=X_a(s)$$

$$or$$

$$\frac{Y_a(s)}{X_a(s)}=\frac{1}{s+0.9}=H(s)$$

$$thus$$

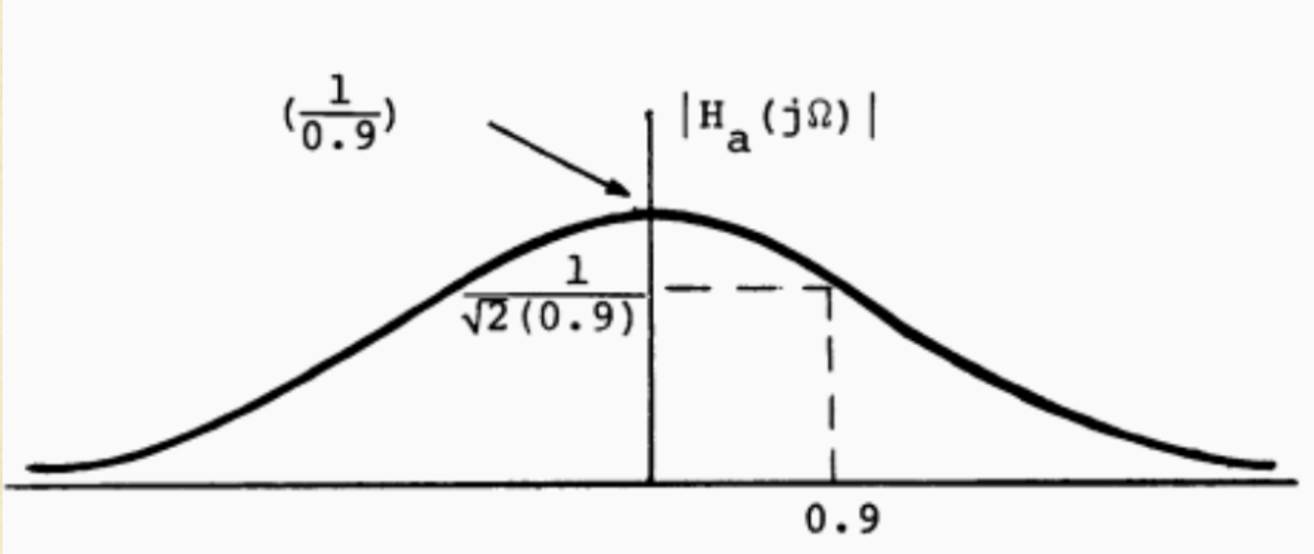
$$H_a(j\Omega)=\frac{1}{j\Omega+0.9}$$

$$and$$

$$\left|H_a(j\Omega)\right|^2=\frac{1}{\Omega^2+(0.9)^2}$$

Example #1 Solution (cont.)

- (a):



- (b): Applying the *z*-transform to both sides of the difference equation we obtain:

$$Y(z)\left[\frac{z-1}{T}\right]+0.9Y(z)=X(z)$$

or

$$H(z)=\frac{Y(z)}{X(z)}=\frac{T}{\left[z+(0.9T-1)\right]}$$

Example #1 Solution (cont.)

- (b): cont.

$$\text{with} \quad T = \frac{10}{9}$$

$$H(z) = \frac{10/9}{z} = \frac{10}{9} z^{-1}$$

and

$$\left| H\left(e^{j\omega}\right) \right| = \frac{10}{9}$$

  - For this particular choice for T the frequency response is constant, independent of frequency in contrast to the analog filter, which is a low pass filter.
  - It emphasizes the fact that the frequency response of the resulting digital filter will in general be severely distorted from that of the original analog filter.

Example #1 Solution (cont.)

- (c): From the system function determined in part (b), the pole is at $z=1-0.9T$. Assuming to be positive, the pole is outside of the unit circle for $T \geq 20/9$.

Example #2

- $h_a(t)$ denotes the impulse response of an analog filter and is given by:

$$h_a(t) = \begin{cases} e^{-0.9t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Let h(n) denote the unit sample response and H(z) denote the system function for the digital filter designed from this analog filter by impulse invariance, i.e. with $h(n)=h_a(nT)$.

- Determine H(z), including T as a parameter and show that for any positive value of T, the digital filter is stable. Indicate also whether the digital filter approximated a low pass filter or a high pass filter.

Example #2 Solution

$$h(n) = h_a(nT) = e^{-0.9nT}u(n)$$

$$= \left(e^{-0.9T}\right)^n u(n)$$

thus

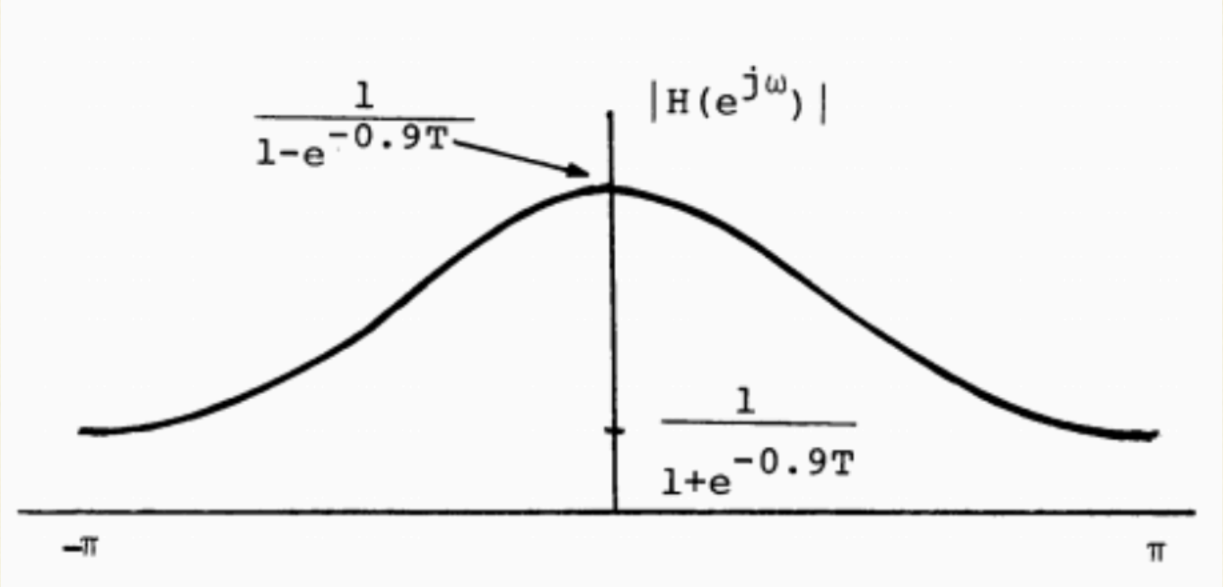
$$H(z) = \frac{1}{1 - e^{-0.9T}z^{-1}}$$

- The frequency response is given by:

$$H\left(e^{j\omega}\right) = \frac{1}{1 - e^{-0.9T}e^{-j\omega}}$$

- The magnitude of which is sketched below:

Example #2 Solution



- The digital filter approximates a low pass filter.
- Since the pole if located at $z=e^{-0.9T}$ the pole is inside the unit circle and hence the system is stable for T>0.