-Digital Signal Processing-
Digital Filter Design-II

Lecture-15
17-May-16

Impulse Invariance

- Sample impulse response of analog filter:

$$h(n) = h_a(nT)$$

$$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left[\frac{j\omega}{T} + \frac{j2\pi k}{T}\right]$$

- Note that aliasing may occur.
- Implementation of digital filter:
  - Partial fraction expansion of analog transfer function (assuming all poles have multiplicity 1)

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

  - Inverse Laplace transform:

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

Impulse Invariance (cont.)

- Sample impulse response:

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} u(n)$$

$$h(n) = \sum_{k=1}^{N} A_k \left(e^{s_k T}\right)^n u(n)$$

- Taking Z-transform:

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

Impulse Invariance (cont.)

- Example:

$$H_a(s) = \frac{(s+a)}{(s+a)^2 + b^2} = \frac{1/2}{s+a+jb} + \frac{1/2}{s+a-jb}$$

$$H(z) = \frac{1/2}{1-e^{-aT}e^{-jbT}z^{-1}} + \frac{1/2}{1-e^{-aT}e^{jbT}z^{-1}}$$

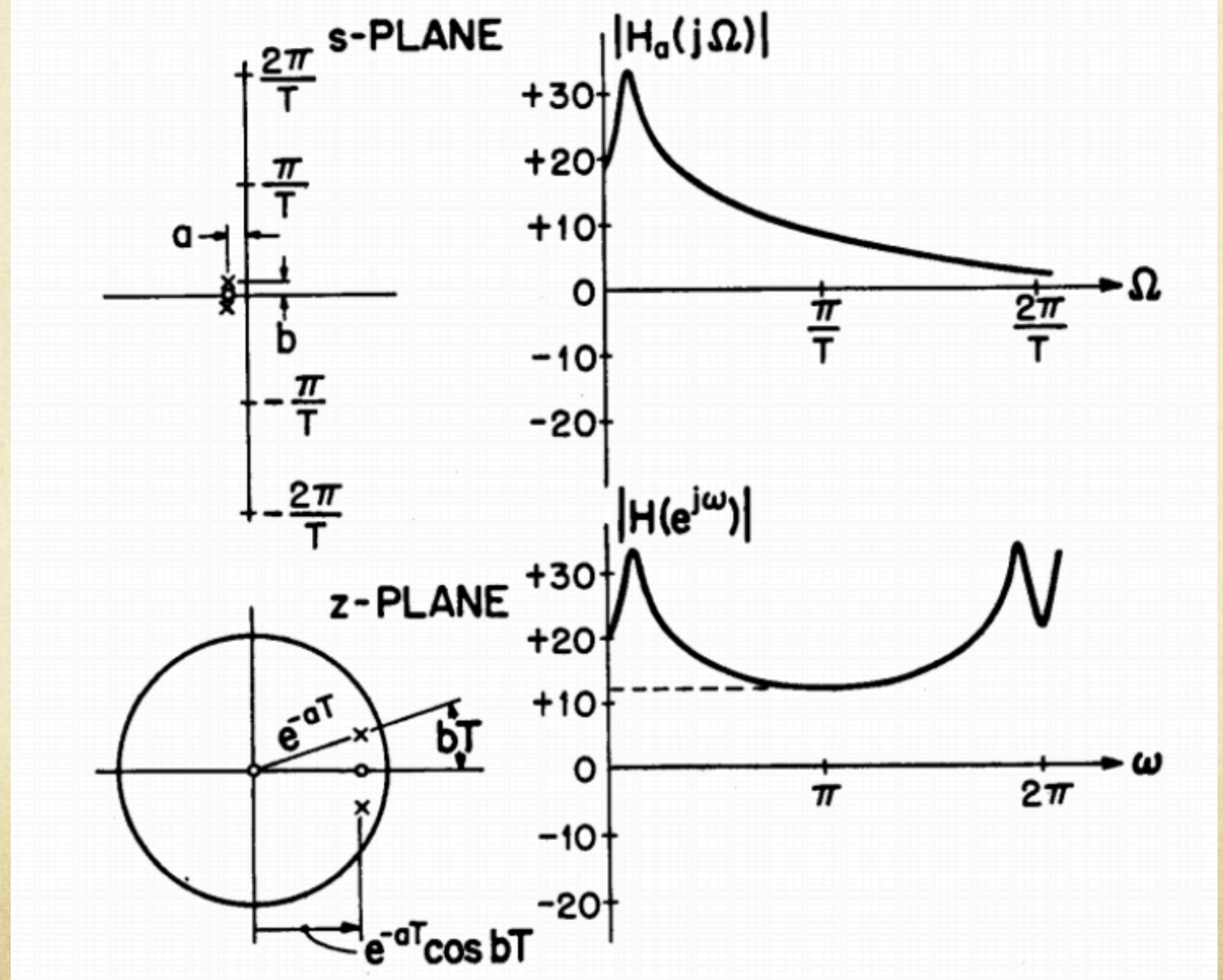
$$= \frac{1-\left(e^{-aT}\cos bT\right)z^{-1}}{\left(1-e^{-aT}e^{-jbT}z^{-1}\right)\left(1-e^{-aT}e^{jbT}z^{-1}\right)}$$

Impulse Invariance Method Summary

- Preserves impulse response and shape of frequency response, if there is no aliasing.
- Desired transition bandwidths map directly between digital and analog frequency domains.
- Pass band and stop band ripple specifications are identical for both digital and analog filters, assuming that there is no aliasing.
- The final digital filter design is independent of the sampling interval parameter T.
- Poles in analog filter map directly to poles in digital filter via transformation.
- There is no such relation between the zeros in the two filters.
- Gain at DC in digital filter may not equal unity, since sampled impulse response may only approximately sum to 1.

Pole-Zero Patterns and Frequency Response

- Pole-zero patterns and frequency response corresponding to the example of viewgraph a:



Bilinear Transformation Method

- This technique avoids the problem of aliasing by mapping $j\Omega$ axis in the s-plane to one revaluation of the unit circle in the *z*-plane.
- Since $-\infty \leq \Omega \leq \infty$ maps onto $-\pi \leq \omega \leq \pi$, the transformation between the continuous-time and discrete-time frequency variables must be non-linear.
- This technique is restricted to situations in which the corresponding warping of the frequency axis is acceptable.
- This method can also be used to design low pass (LP), high pass (HP), band pass (BP) and band stop (BS), Butterworth, Chebyshev, Inverse-Chebyshev and Elliptic filters.
- If $H_a(s)$ is the continues time transfer function the discrete time transfer function is detained by replacing s with:

$$H_a(s) \Rightarrow H(z)$$

$$s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$$

$$z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

Bilinear Transformation Method (cont.)

$$for \quad z = e^{j\omega} \quad e^{-j\frac{\omega}{2}}\left[e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]$$

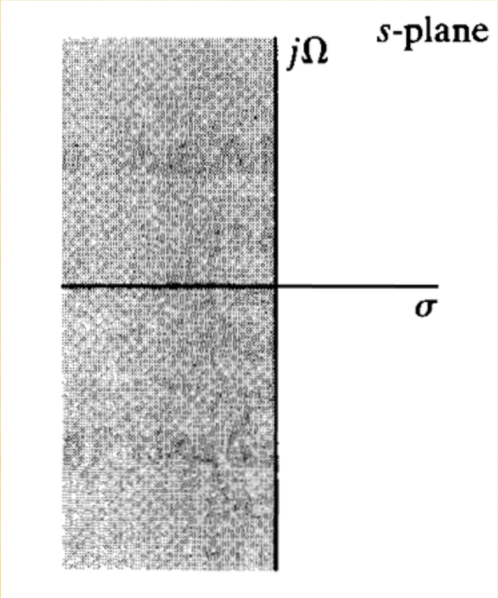
$$s = \frac{2}{T}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] = \sigma + j\Omega = \frac{2}{T}\left[\frac{2e^{-j\frac{\omega}{2}} j\sin(\omega/2)}{2e^{-j\frac{\omega}{2}}\cos(\omega/2)}\right] = \frac{2j}{T}\tan\frac{\omega}{2}$$

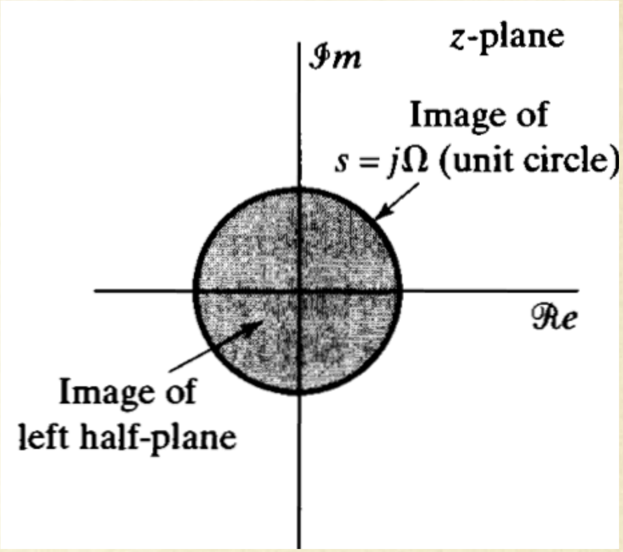
which yields

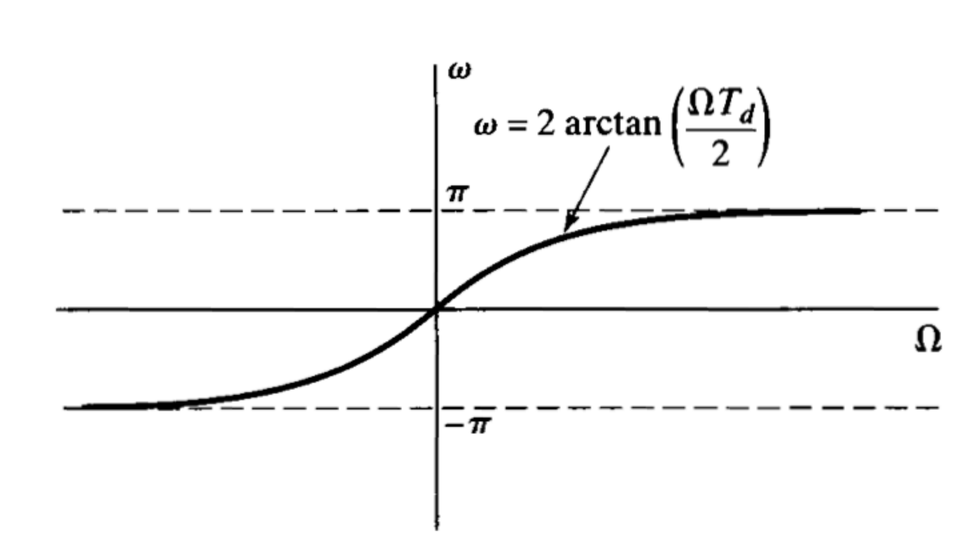
$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad or \quad \omega = 2\arctan\left(\frac{\Omega T}{2}\right)$$

$$j\Omega \quad axis \quad \Leftrightarrow \quad unit \quad circle$$

Bilinear Transformation Method (cont.)

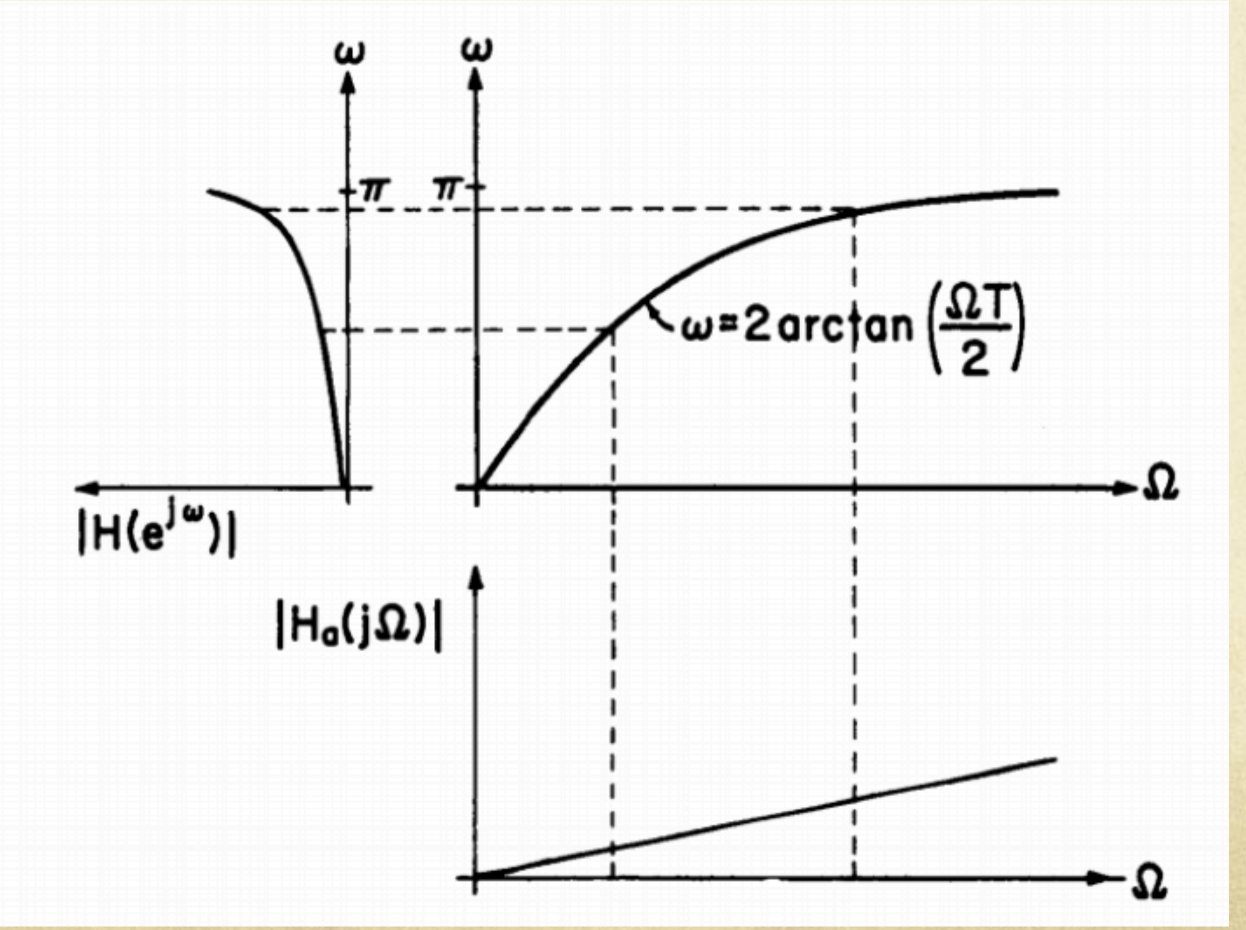






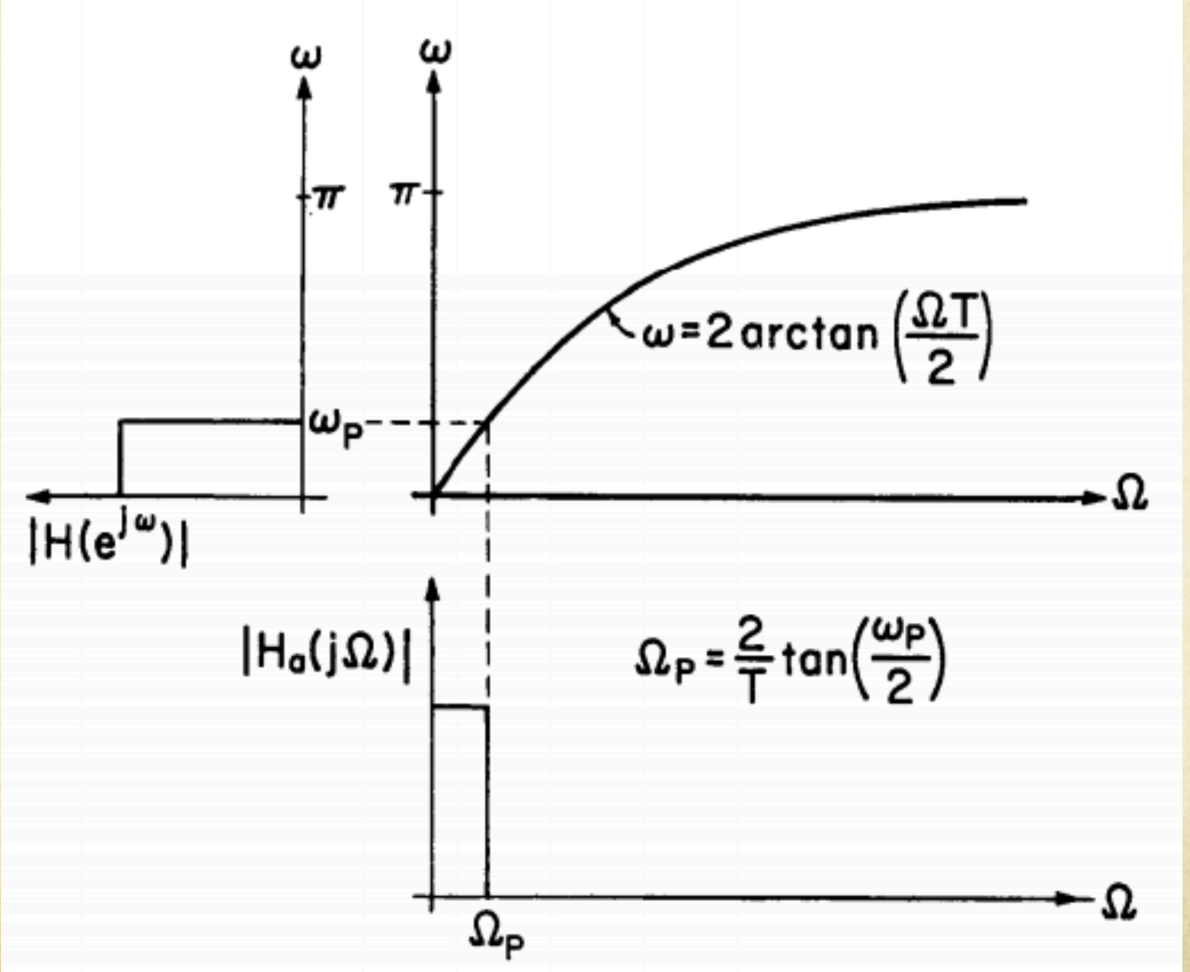
Effect of Bilinear Transformation

- Illustration of effect of bilinear transformation on a piece-wise constant frequency response characteristic:



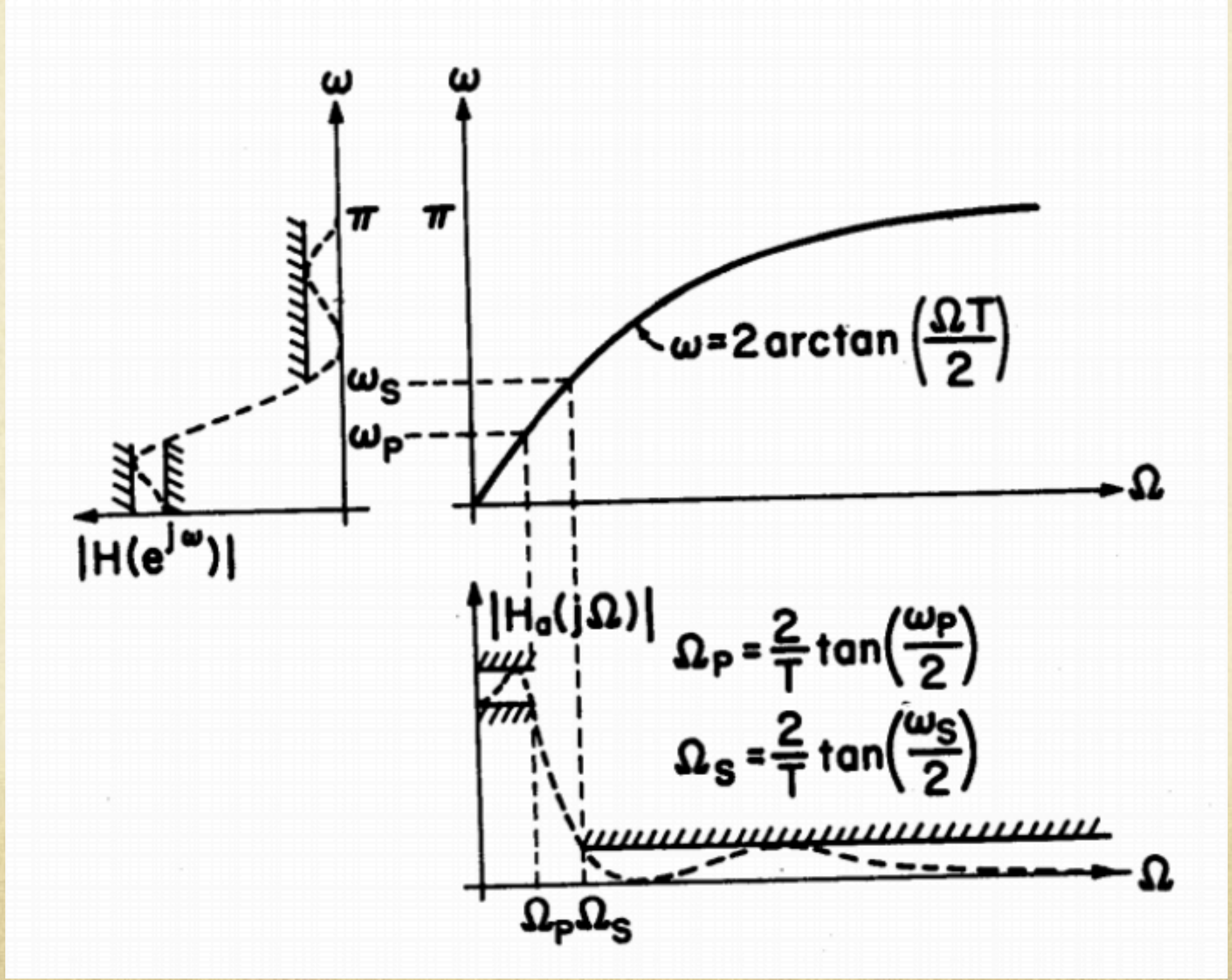
Effect of Bilinear Transformation

- Illustration of effect of bilinear transformation on an equi-ripple frequency response characteristics:



Effect of Frequency Warping

- Illustration of effect of frequency warping inherent in the bilinear transformation is:



Frequency Selective Filters

- Typical frequency-selective continuous-time approximation are:
  - Butterworth
  - Chebyshev
  - Elliptic Filters

Butterworth Filter

- The Butterworth filter of order n is described by the magnitude square frequency response of:

$$\left|H_n(j\Omega)\right|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}$$

- It has the following properties:
  - $|H_n(j\Omega)|^2=1$ at $\Omega=0$
  - $|H_n(j\Omega)|^2=1/2$ at $\Omega=\Omega_c$
  - $|H_n(j\Omega)|^2$ is monotonically decreasing function of Ω.
  - As n gets larger, $|H_n(j\Omega)|^2$ approaches an ideal low pass filter.
  - $|H_n(j\Omega)|^2$ is called maximally flat at origin, since all order derivative exist and they are zero at $\Omega=0$.

- The poles of a Butterworth filter lie on circle of radius Ω_c in s-plane.

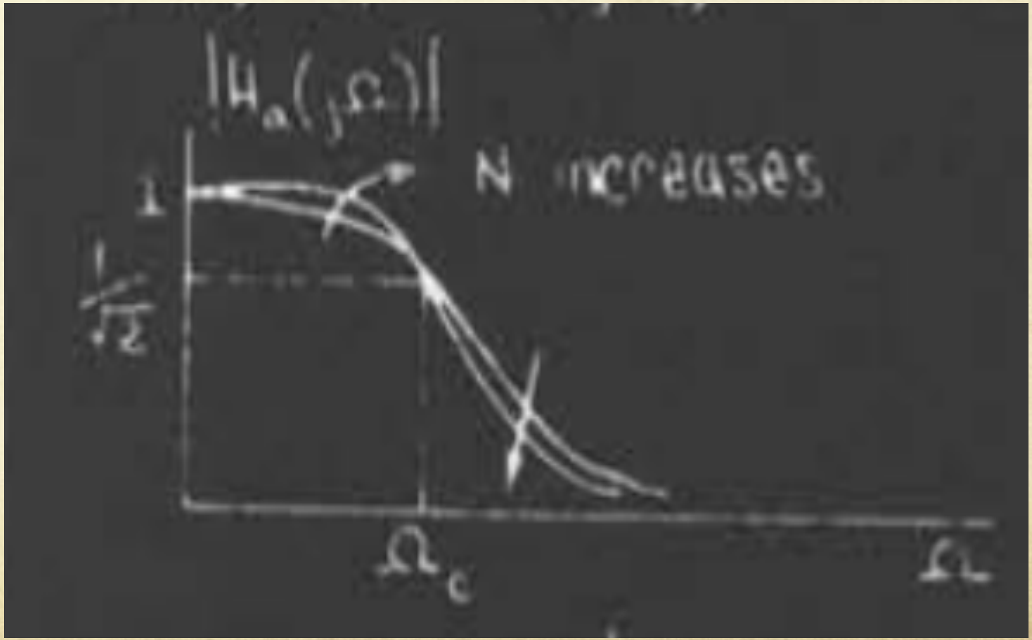
Butterworth Filter (cont.)

Analog Butterworth Filter: $$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$where \quad \Omega_c \rightarrow \quad cutoff \quad frequency$$

$$N \rightarrow \quad order \quad of \quad filter$$

- N effects the shape of frequency response.
  - If N is larger, the frequency response tends to be flatter longer and drop of shorter and vice versa.
  - The higher the order of the filter the sharper the drop from pass band to stop band region.



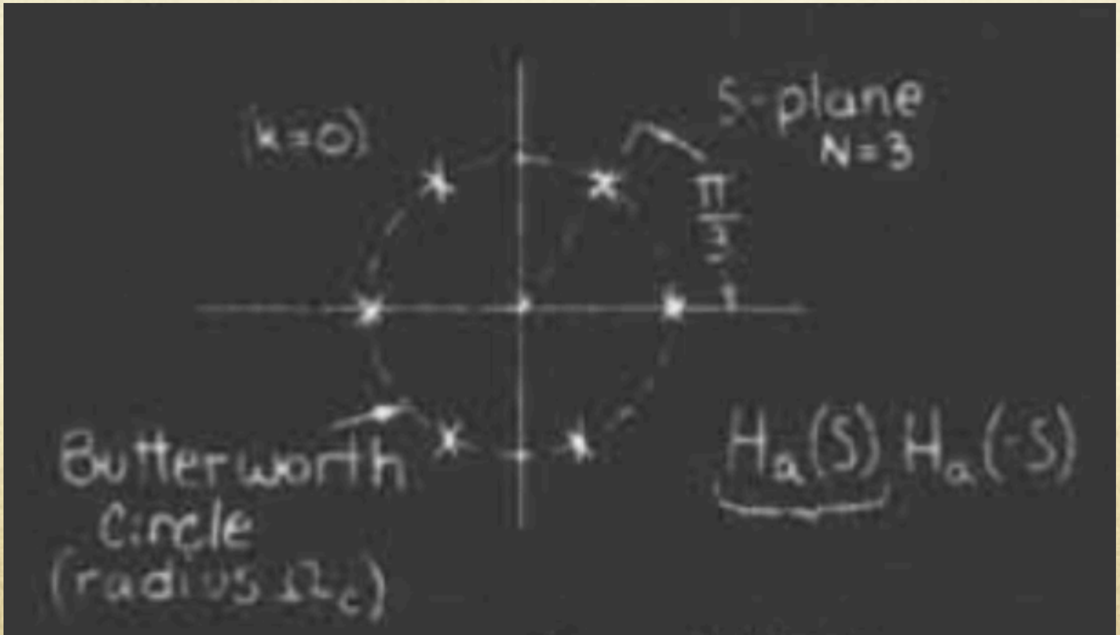
Butterworth Filter (cont.)

$$H_a(S)H_a(-S) = \frac{1}{1+\left(\frac{S}{j\Omega_c}\right)^{2N}}$$

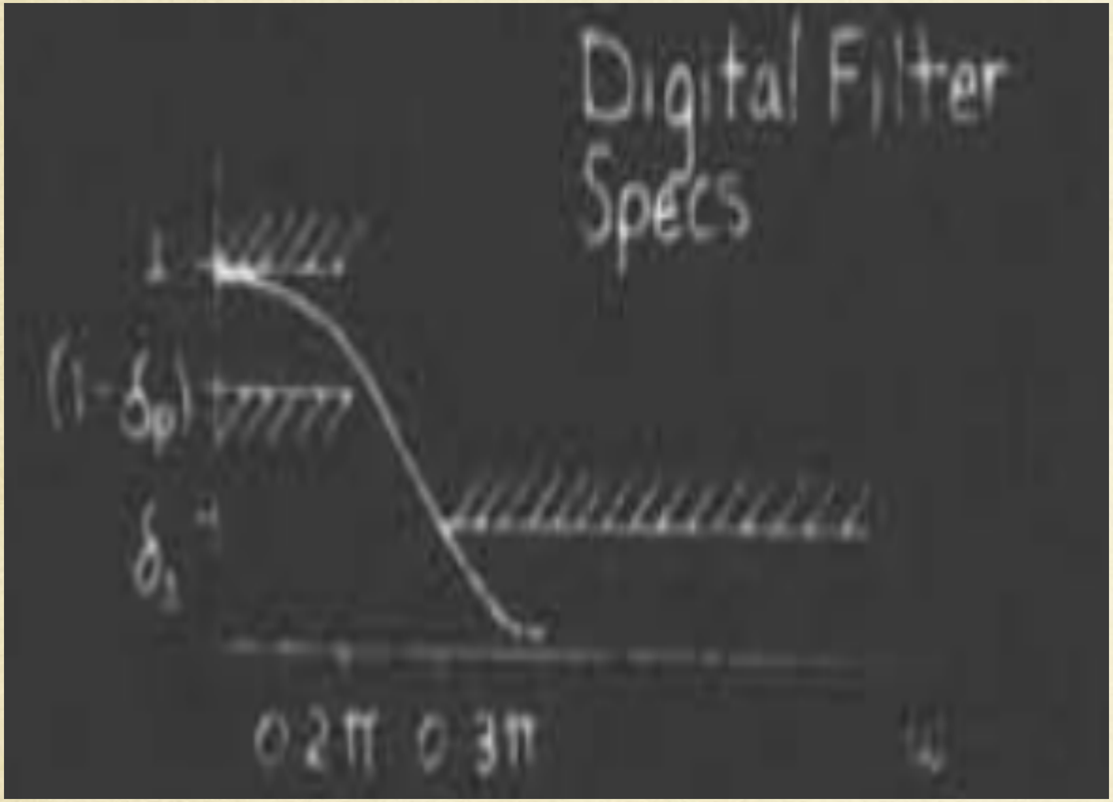
- Poles at:

$$S_p = (-1)^{1/2N}(j\Omega_c)$$

$$S_p = e^{j\left[\frac{\pi+2k\pi}{2N}\right]}e^{j\frac{\pi}{2}}\Omega_c$$



Butterworth Filter (cont.)



$$\left(1-\delta_p\right) \geq -1db$$

$$\delta_s \leq -15db$$

$$20\log_{10}\left|H\left(e^{j0.2\pi}\right)\right| \geq -1$$

$$or \quad \left|H\left(e^{j0.2\pi}\right)\right| \geq 10^{-.05}$$

also

$$20\log_{10}\left|H\left(e^{j0.3\pi}\right)\right| \leq -15$$

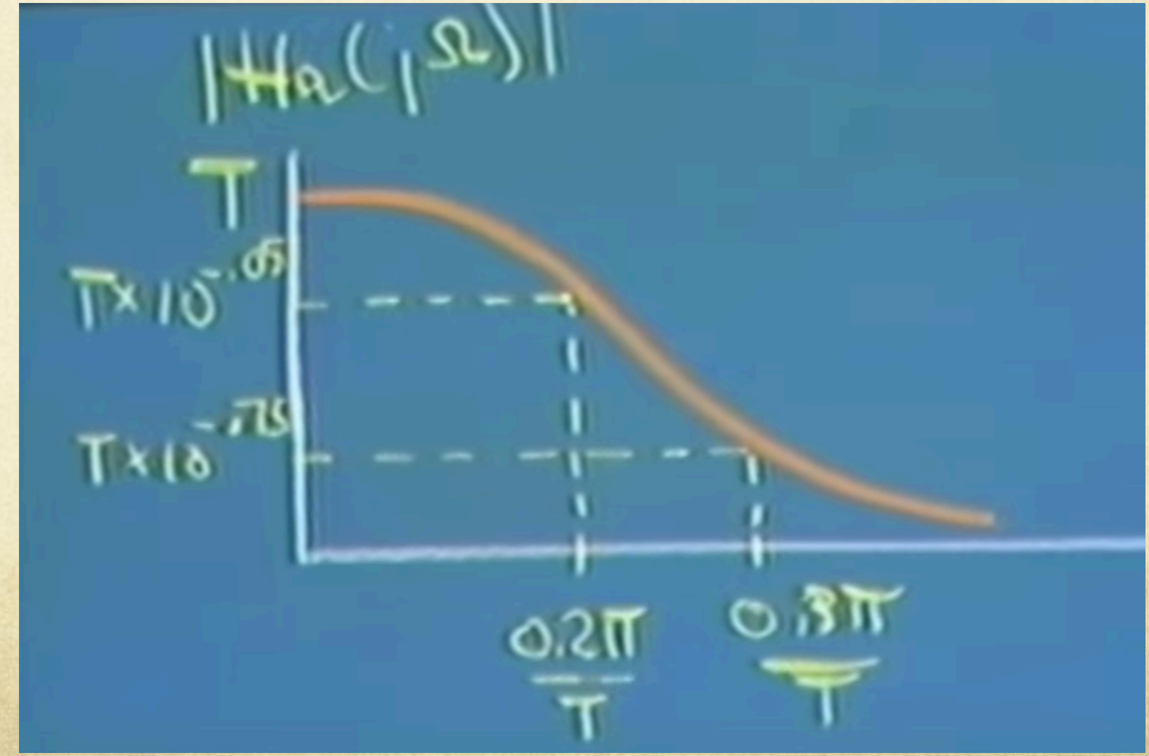
$$or \quad \left|H\left(e^{j0.3\pi}\right)\right| \leq 10^{-.75}$$

Impulse Invariant Design

$$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left[\frac{j\omega}{T} + j\frac{2\pi k}{T}\right]$$

$$\Omega = \frac{\omega}{T}$$

- Neglect aliasing.



Impulse Invariant Design (cont.)

$$\left|H_a\left(j\Omega\right)\right|^2 = \frac{T^2}{1+\left(\dfrac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{j\dfrac{0.2\pi}{T}}{j\Omega_c}\right)^{2N} = 10^{.1} \rightarrow (1)$$

$$1+\left(\frac{j\dfrac{0.3\pi}{T}}{j\Omega_c}\right)^{2N} = 10^{1.5} \rightarrow (2)$$

$$N = (5.8858)$$

$$\Omega_c T = .70474$$

Impulse Invariant Design (cont.)

- N should be an integer so we round up N , that is N=6.

$$1+\left(\frac{j\frac{0.2\pi}{T}}{j\Omega_c}\right)^{2\times 6}=10^{-1}$$

$$\Omega_c T = 0.7032$$

$$H_a(S)H_a(-S)=\frac{T^2}{1+\left(\frac{ST}{j.7032}\right)^{2\times 6}}$$

$$T=1$$

$$h_a(t)\leftrightarrow H_a(S)$$

$$h_a\left(\frac{t}{T}\right)\leftrightarrow TH_a(ST)$$

$$h(n)=h_a(nT)=h_a(n)$$

Impulse Invariant Design (cont.)

- The LHP poles are:
  - Pole pair 1:

$$-0.182 \pm j(0.679)$$

  - Pole pair 2:

$$-0.497 \pm j(0.497)$$

  - Pole pair 3:

$$-0.679 \pm j(0.182)$$

Impulse Invariant Design (cont.)

- Poles of transfer function:

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

- The transfer function:

$$H(s) = \frac{0.12093}{(s^2 + 0.364s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

- Mapping to *z*-domain:

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.257z^{-2}}$$