-Digital Signal Processing-
Digital Filter Design-III

Lecture-16
18-May-16

Bilinear Transform Design

- Bilinear transform applied to Butterworth:

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

- Apply bilinear transformation to specifications:

$$0.89125 \le \left|H(j\Omega)\right| \le 1 \qquad 0 \le |\Omega| \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$$

$$\left|H(j\Omega)\right| \le 0.17783 \qquad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| < \infty$$

- We can assume T_d=1 and apply the specifications to:

$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}}$$

- To get:

$$1 + \left(\frac{2\tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1 + \left(\frac{2\tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

Bilinear Transform Design (cont.)

- Solve N and Ω_c

$$N = \frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) \Big/ \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2\log\left[\tan(0.15\pi)/\tan(0.1\pi)\right]} = 5.305 \cong 6 \qquad \Omega_c = 0.766$$

- The resulting transfer function has the following poles

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\dots,11$$

- Resulting in

$$H_c(s) = \frac{0.20238}{\left(s^2 + 0.3996s + 0.5871\right)\left(s^2 + 1.0836s + 0.5871\right)\left(s^2 + 1.4802s + 0.5871\right)}$$

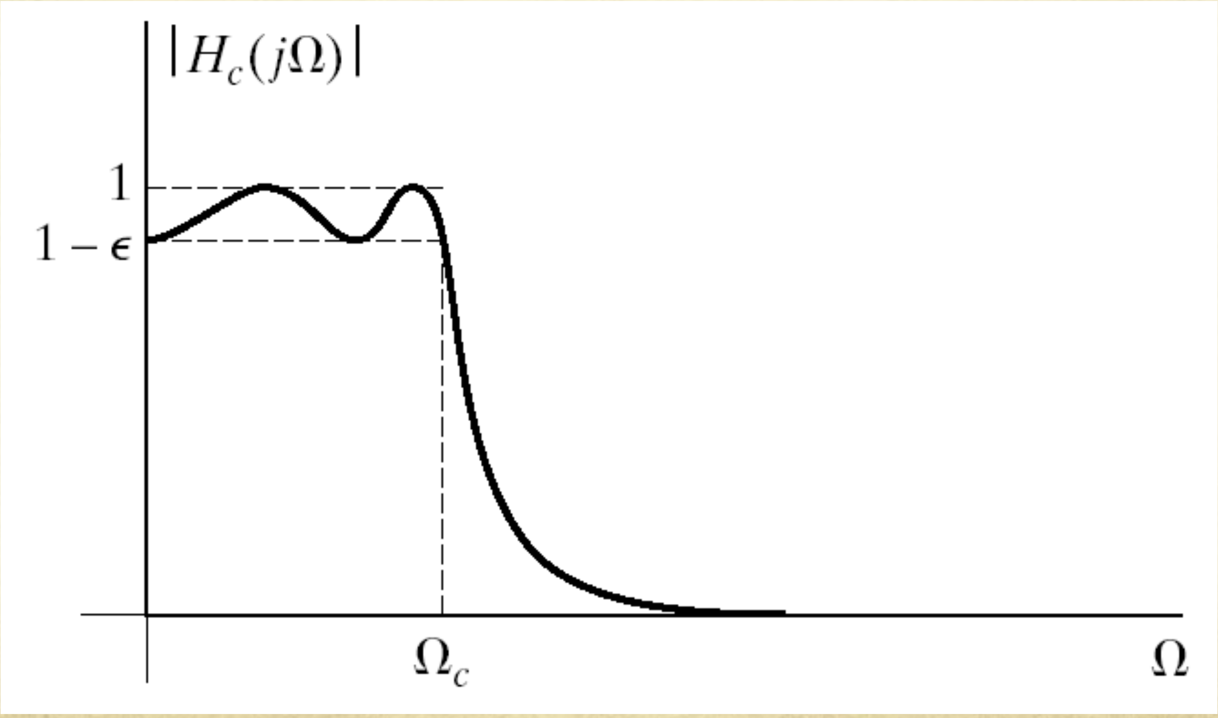
- Applying the bilinear transform yields

$$H(z) = \frac{0.0007378\left(1 + z^{-1}\right)^6}{\left(1 - 1.2686z^{-1} + 0.7051z^{-2}\right)\left(1 - 1.0106z^{-1} + 0.3583z^{-2}\right)} \times \frac{1}{\left(1 - 0.9044z^{-1} + 0.2155z^{-2}\right)}$$

Chebyshev Filter

- Equiripple in the pass band and monotonic in the stop band.
- Or equiripple in the stop band and monotonic in the pass band.

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 V_N^2(\Omega / \Omega_c)} \qquad V_N(x) = \cos(N \cos^{-1} x)$$



Example #1

- The system function of a discrete-time system is:

$$H(z) = \frac{2}{1 - e^{-0.2}z^{-1}} - \frac{1}{1 - e^{-0.4}z^{-1}}$$

- (a): Assume that this discrete time filter was designed by the impulse invariance method with T_d=2; i.e., h[n]=$2h_c(2n)$, where $h_c(t)$ is real. Find the system function $H_c(s)$ of a continuous time filter that could have been the basis for the design. Is your answer unique? If not find another system function $H_c(s)$.
- (b): Assume that H(z) was obtained by the bilinear transform method with T_d=2 . Find the system function $H_c(s)$ that could have been the basis for the design. Is your answer unique? If not find another $H_c(s)$.

Example #1 Solution

- (a): In the impulse invariance design the poles transform as: $z_k = e^{sT}$ and we have the relationship:

$$\frac{1}{s+a} \leftrightarrow \frac{T_d}{1-e^{-aT_d}z^{-1}}$$

- Therefore,

$$H_c(s) = \frac{2/T_d}{s+0.2} - \frac{1/T_d}{s+0.4}$$

$$= \frac{1}{s+0.2} - \frac{0.5}{s+0.4}$$

- The above solution is not unique due to the periodicity of $z=e^{j\omega}$. A more general answer is:

$$H_c(s) = \frac{2/T_d}{s+\left(0.2+j\frac{2\pi k}{T_d}\right)} - \frac{1/T_d}{s+\left(0.4+j\frac{2\pi l}{T_d}\right)}$$

$$where \quad k \quad \& \quad l \quad are \quad integers$$

Example #1 Solution (cont.)

- (b): Using the inverse relationship for the bilinear transform:

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$

- We get:

$$H_c(s) = \frac{2}{1-e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1-e^{-0.4}\left(\frac{1-s}{1+s}\right)}$$

$$= \frac{2(s+1)}{s(1+e^{-0.2})+(1-e^{-0.2})} - \frac{(s+1)}{s(1+e^{-0.4})+(1-e^{-0.4})}$$

$$= \left(\frac{2}{1+e^{-0.2}}\right)\left(\frac{s+1}{s+\frac{1-e^{-0.2}}{1+e^{-0.2}}}\right) - \left(\frac{1}{1+e^{-0.4}}\right)\left(\frac{s+1}{s+\frac{1-e^{-0.4}}{1+e^{-0.4}}}\right)$$

- Since the bilinear transform does not introduce any ambiguity, the representation is unique.

Overview of IIR Filter Design

- IIR digital filter designs are based on established methods for designing analog filters.
- Approach is generally limited to frequency selective filters with ideal pass-band/stop-band characteristics.
- Basic filter type is low pass.
- Achieve high pass or band pass via transformations.
- Achieve multiple stop/pass bands by combining multiple filters with single pass band.

IIR Filter Design Steps

- Choose prototype analog filter family:
  - Butterworth
  - Chebyshev Type I or II
  - Elliptic
- Choose analog-digital transformation method:
  - Impulse invariance
  - Bilinear transformation
- Transform digital filter specifications to equivalent analog filter specifications.
- Design analog filter.
- Transform analog filter to digital filter.
- Perform frequency transformations to achieve high pass or band pass filter if desired.

Example #2

- Let $|H(j\Omega)|^2$ denote the squared magnitude function for an analog Butterworth filter of order 5 with a cutoff frequency Ω_c of $2\pi\times10^3$. Determine and indicate in the s-plane the poles of the system function H(s). Assume that the system is stable and causal.

Example #2 Solution

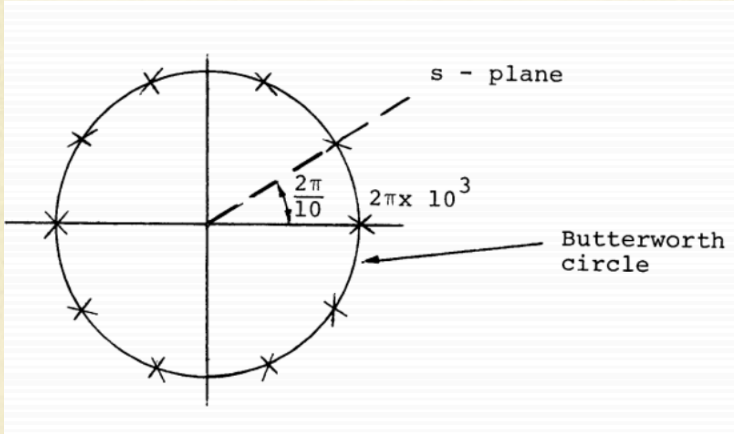
- The squared magnitude function for a fifth order Butterworth filter with cutoff frequency $\Omega_c = 2\pi \times 10^3$ is given by:

$$H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j2\pi \times 10^3} \right)^{10}}$$

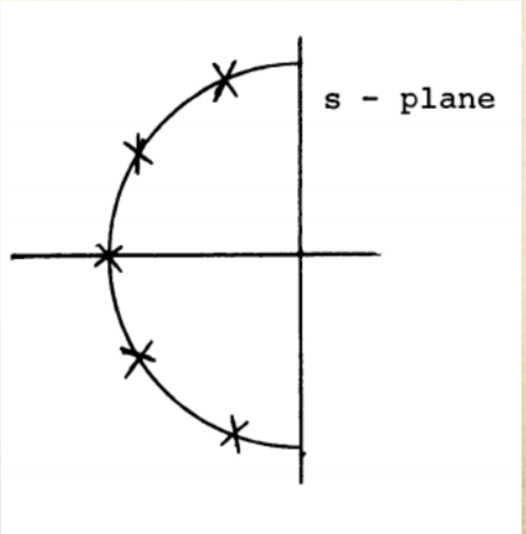
- The poles of H(s)H(-s) are the roots of $1 + \left(\frac{s}{j2\pi \times 10^3} \right)^{10} = 0$
- Or

$$s = (-1)^{1/10} \left(j2\pi \times 10^3 \right)$$

Example #2 Solution



- Since H(s) corresponds to a stable, causal filter, we factor the squared magnitude function so that the left-half plane poles correspond to H(s) and the right-half plane poles correspond to H(-s).
- Thus the poles of H(s) are indicated below:



Example #3

- Design a first order digital low pass filter with a 3dB cutoff frequency of ω_c =0.25π by applying the bilinear transformation to the analog Butterworth filter:

$$H_a(s) = \frac{1}{1 + s/\Omega_c}$$

- Because the 3-dB cutoff frequency of the Butterworth filter is Ω_c, for a cutoff frequency ω_c =0.25π in the digital filter, we must have:

$$\Omega_c = \frac{2}{T_d}\tan\left(\frac{0.25\pi}{2}\right) = \frac{0.828}{T_d}$$

- Therefore, the system function of the analog filter is:

$$H_a(s) = \frac{1}{1 + sT_d/0.828}$$

- Applying the bilinear transformation to the analog filter gives:

$$H[z] = H_a(s)\Big|_{s=\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{1 + (2/0.828)\left[\left(1-z^{-1}\right)/\left(1+z^{-1}\right)\right]}$$

$$H[z] = 0.2920\frac{1+z^{-1}}{1-0.4159z^{-1}}$$

- Note that the parameter T_d does not enter into the design.