-Digital Signal Processing- FIR Filter Design

Lecture-17
24-May-16

FIR Filter Design

- FIR filters can also be designed from a frequency response specification.
- The equivalent sampled impulse response which determines the coefficients of the FIR filter can then be found by inverse Discrete Fourier transformation.
- The frequency response of an Nth-order causal FIR filter is:

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{N}h\left(n\right)e^{-jn\omega}$$

- The design of an FIR filter involves finding the coefficients h(n) that result in a frequency response that satisfies a given set of filter specifications.

FIR Filter Design (cont.)

- FIR filter have two important advantages over IIR filters:
  - They are guaranteed to be stable, even after the filter coefficients have been quantized.
  - They may be easily constrained to have linear phase.

Basic Design Methods

- Windows
- Frequency Sampling
- Equiripple Design

Filter Design by Windowing

- Simplest way of designing FIR filters
- Method is all discrete-time no continuous-time involved
- Start with ideal frequency response

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

- Choose ideal frequency response as desired response
- Most ideal impulse responses are of infinite length
- The easiest way to obtain a causal FIR filter from ideal is

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

- More generally

$$h[n] = h_d[n]w[n] \quad \text{where} \quad w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Windowing in Frequency Domain

- Windowed frequency response:

$$H\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

- Thus, the ideal frequency response is smoothed by the discrete time Fourier transform of the window, $W(e^{j\omega})$.

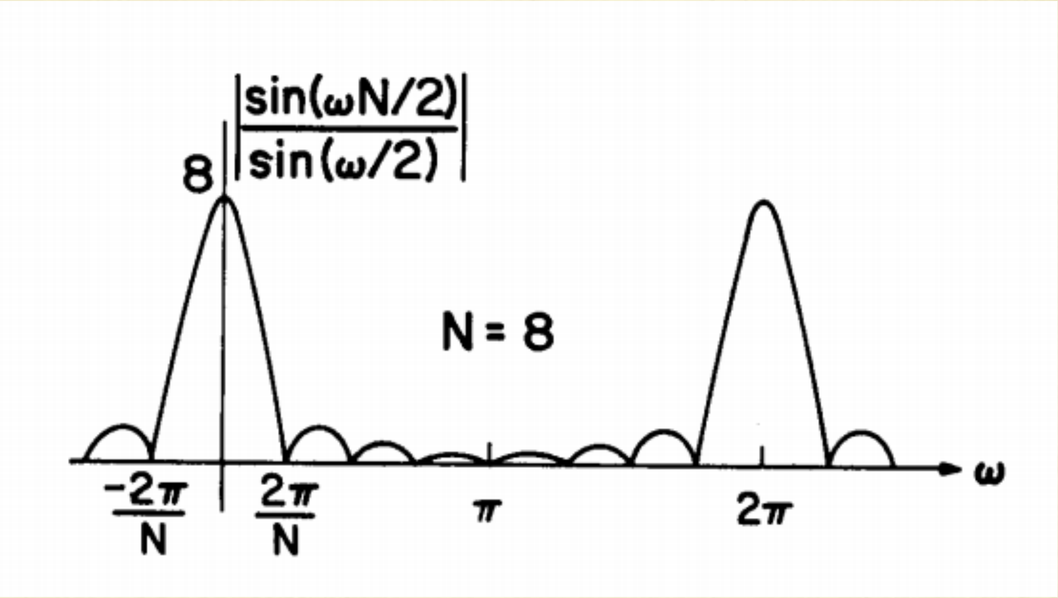
Windowing in Frequency Domain (cont.)

- There are many different types of windows that may be used in the window design method, a few of which are listed below:

Rectangular	$w(n)=\begin{cases} 1 & 0 \le n \le N \\ 0 & else \end{cases}$
Hanning or Hann window	$w(n)=\begin{cases} 0.5-0.5\cos\left(\frac{2\pi n}{N}\right) & 0 \le n \le N \\ 0 & else \end{cases}$
Hamming	$w(n)=\begin{cases} 0.54-0.46\cos\left(\frac{2\pi n}{N}\right) & 0 \le n \le N \\ 0 & else \end{cases}$
Blackman	$w(n)=\begin{cases} 0.42-0.5\cos\left(\frac{2\pi n}{N}\right)+0.08\cos\left(\frac{4\pi n}{N}\right) & 0 \le n \le N \\ 0 & else \end{cases}$

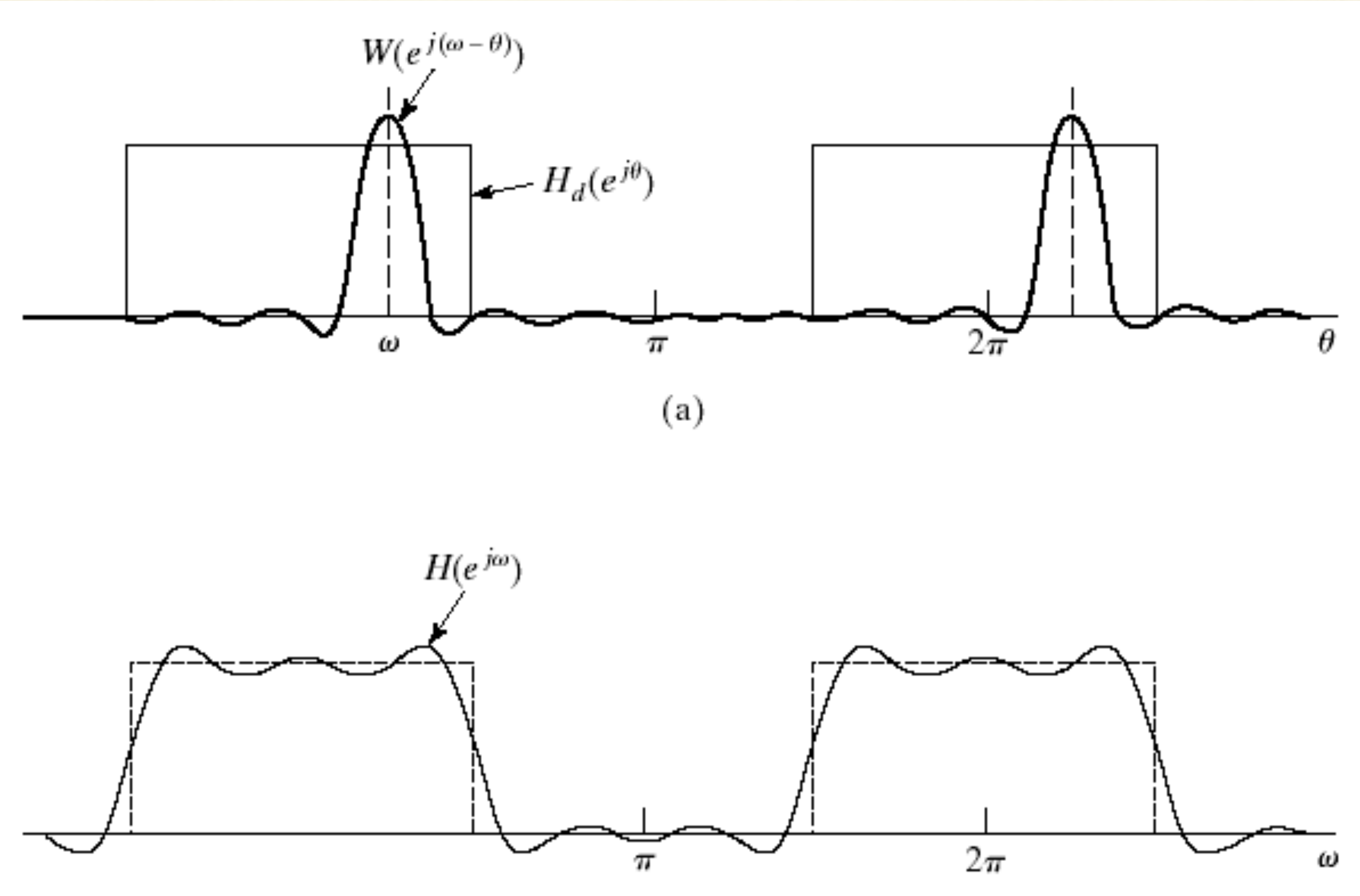
Windowing in Frequency Domain (cont.)

- Magnitude of the Fourier transform for an eight point rectangular window:



Windowing in Frequency Domain (cont.)

- The windowed version is smeared version of desired response.



- If w[n]=1 for all n, then $W(e^{j\omega})$ is pulse train with 2π period.

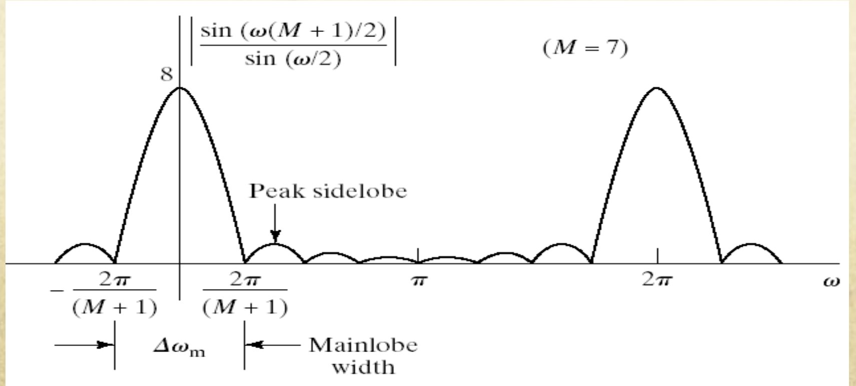
Windowing in Frequency Domain (cont.)

- There are two methods with which we can determine the desired frequency response :
  - The width of the main lobe of $W(e^{j\omega})$.
- Ideally the main lobe should be narrow and the side lobe amplitude should be small .
- However for fixed length window these cannot be minimized independently.
- General properties of windows are as follows:
  - As the length N of the window increases, the width of the main lobe decreases, which results in a decrease in the transition width between pass bands and stop bands. This relationship is given approximately by $N\Delta f=c$.
  - The peak side lobe amplitude of the window is determined by the shape of the window and it is essentially independent of the window length.
  - If the window shape is changed to decrease the side lobe amplitude, the width of the main lobe will generally increase.

Properties of Windows

- Prefer windows that concentrate around DC in frequency
  - Less smearing, closer approximation
- Prefer window that has minimal span in time
  - Less coefficient in designed filter, computationally efficient
- So we want concentration in time and in frequency
  - Contradictory requirements
- Example: Rectangular window

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = e^{-j\omega M/2} \frac{\sin[\omega(M+1)/2]}{\sin[\omega/2]}$$

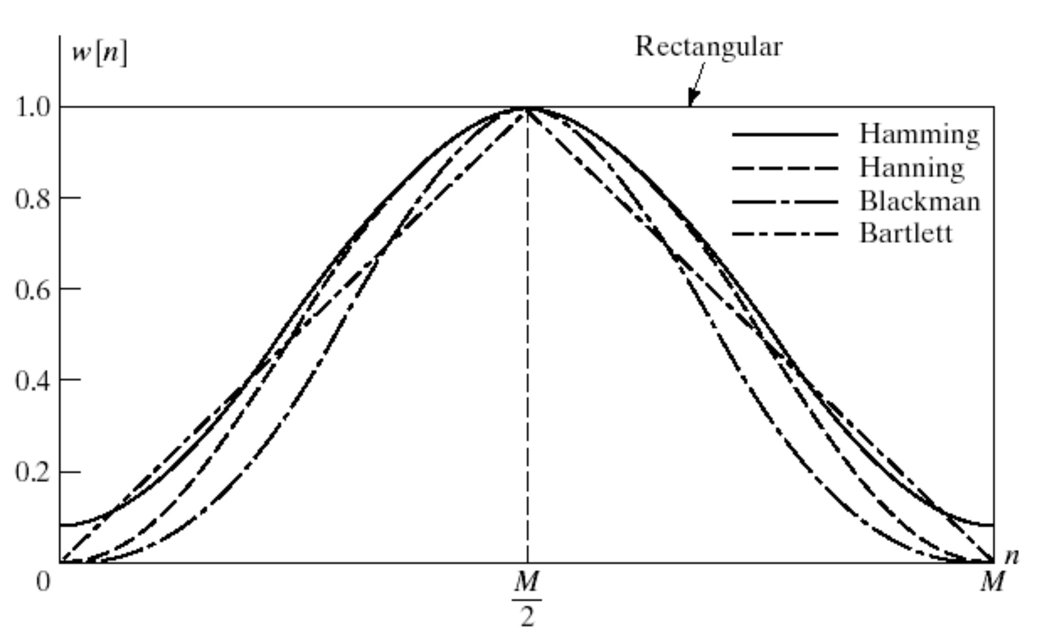


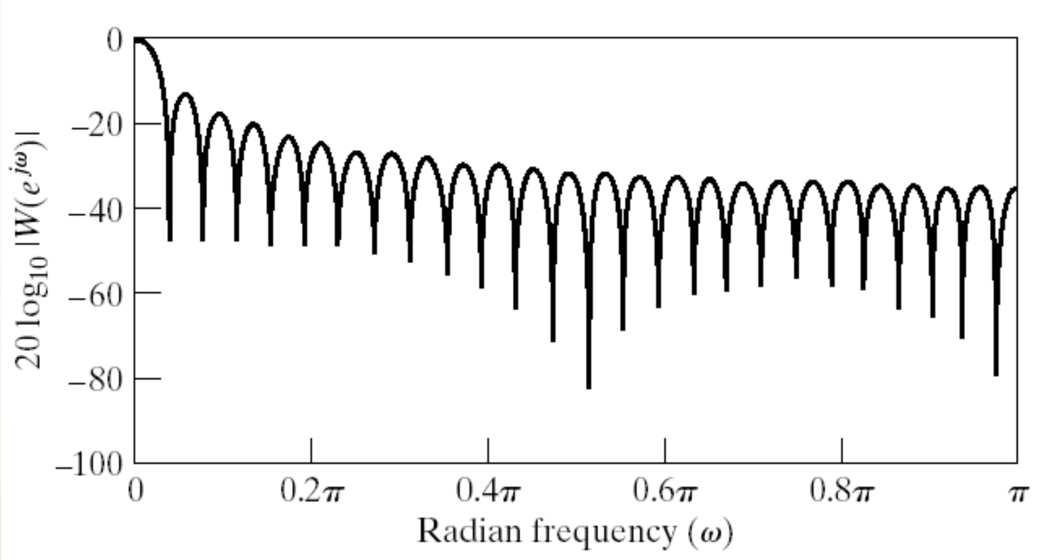
Rectangular Window

- Narrowest main lobe
  - $4\pi/(M+1)$
  - Sharpest transitions at discontinuities in frequency
- Large side lobes
  - -13 dB
  - Large oscillation around discontinuities
- Simplest window possible:

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Rectangular Window (cont.)



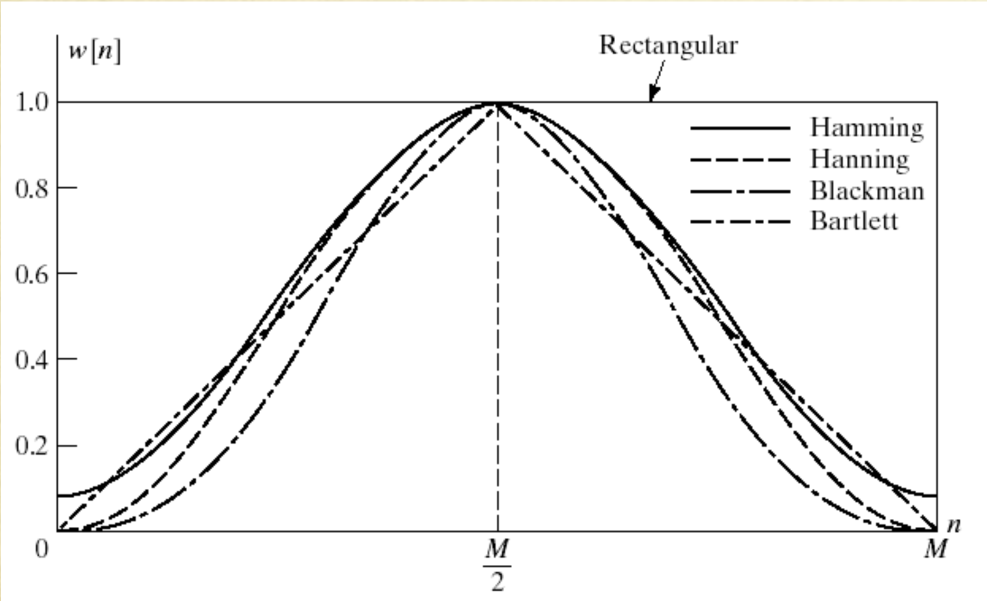


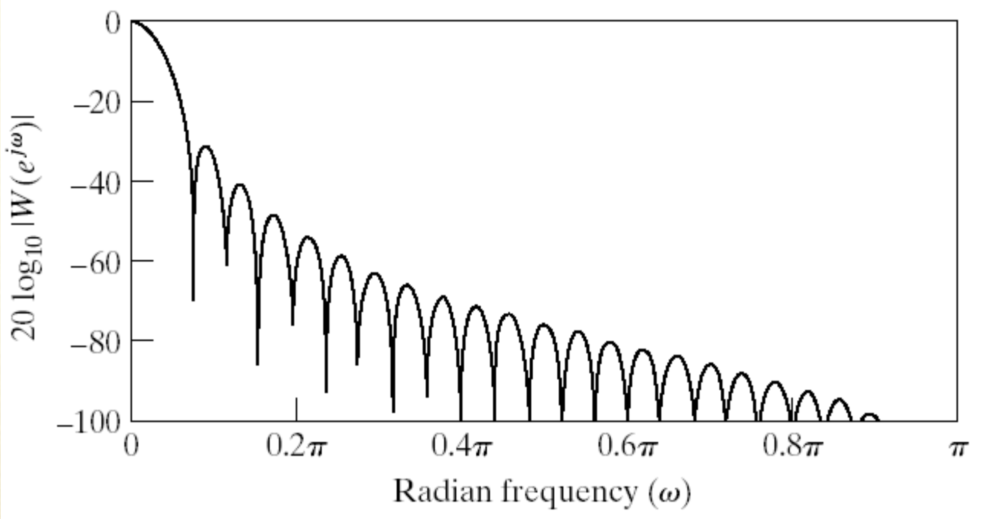
Hanning Window

- Medium main lobe
  - $8\pi/M$
- Side lobes
  - -31 dB
- Hamming window performs better
- Same complexity as Hamming

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Hanning Window (cont.)



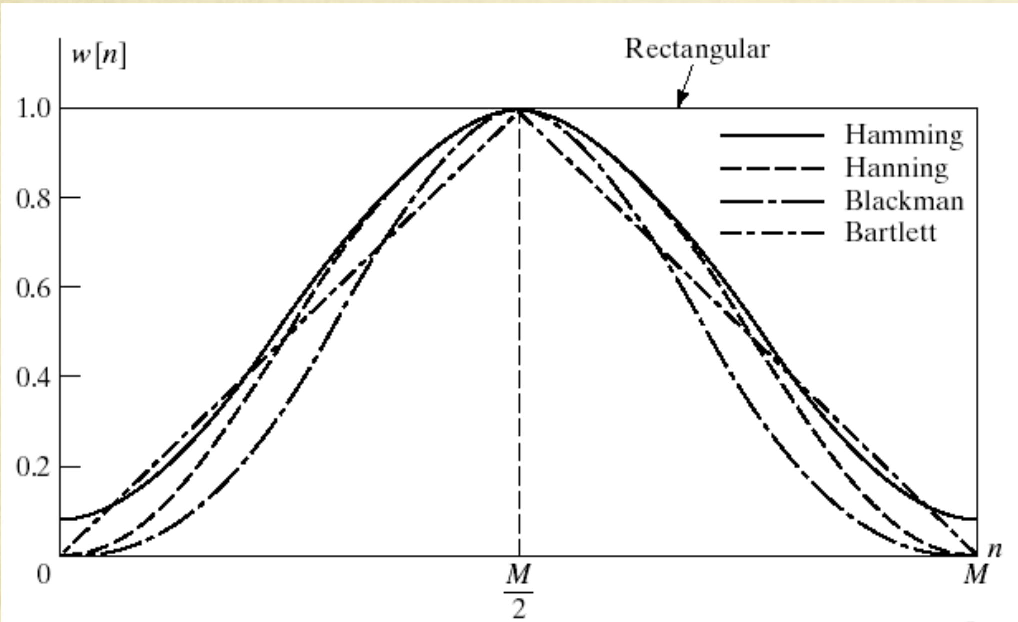


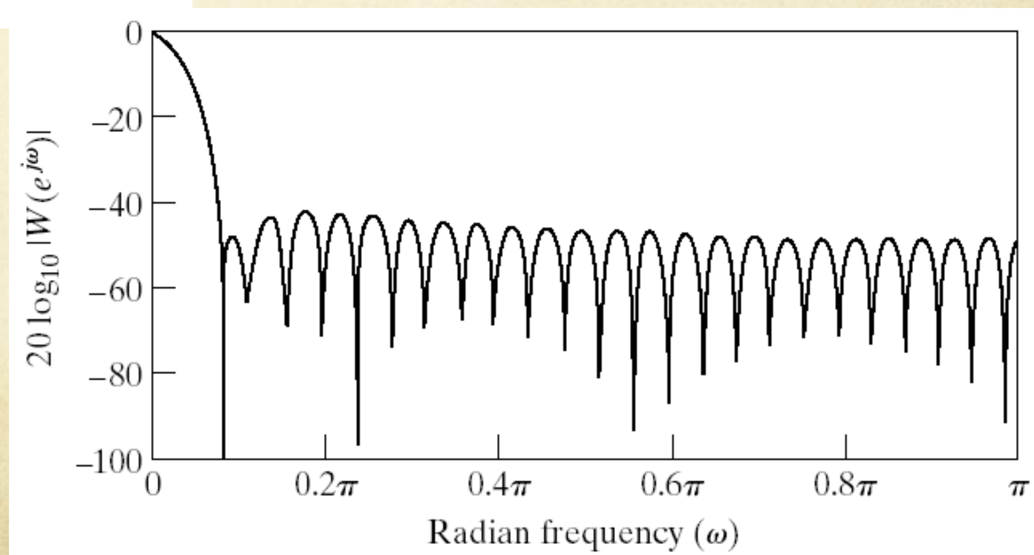
Hamming Window

- Medium main lobe
  - $8\pi/M$
- Good side lobes
  - -41 dB
- Simpler than Blackman

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Hamming Window (cont.)



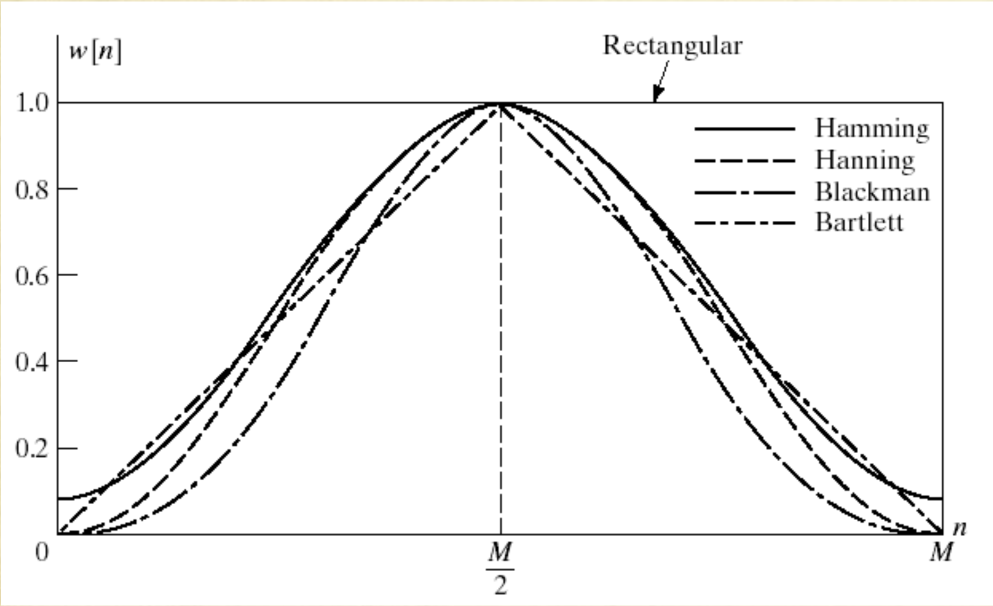


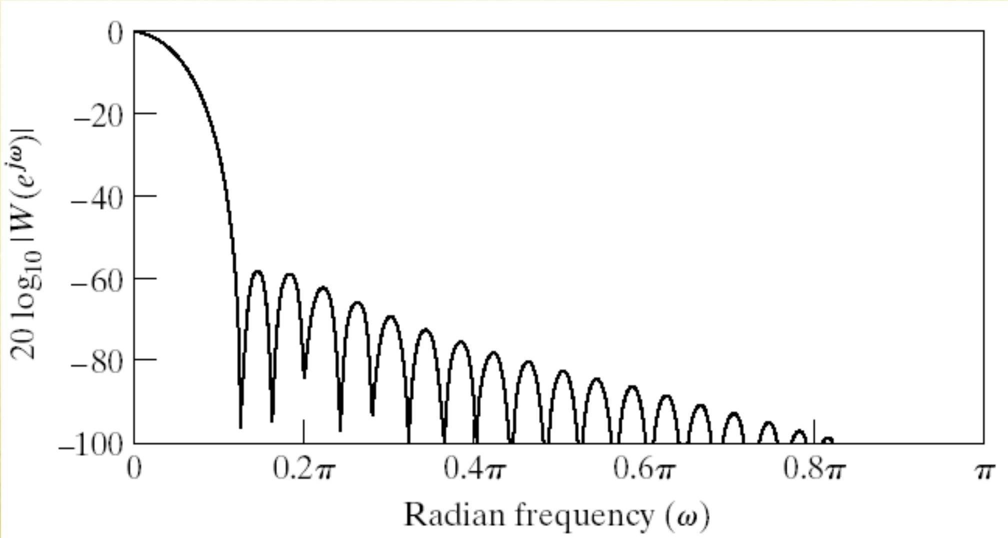
Blackman Window

- Large main lobe
  - $12\pi/M$
- Very good side lobes
  - -57 dB
- Complex equation:

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Blackman Window (cont.)





Example # 1

- Suppose that we would like to design an FIR linear phase low pass filter according to the following specifications:

$$0.99 \le \left|H\left(e^{j\omega}\right)\right| \le 1.01 \quad 0 \le |\omega| \le 0.19\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.01 \quad 0.21\pi \le |\omega| \le \pi$$

- For a stop band attenuation of 20 log (0.01)= -40dB, we may use the Hamming window.

- Because the specification calls for a transition width of:

$$\Delta\omega = \omega_s - \omega_p = 0.02\pi$$

$$or$$

$$\Delta f = 0.01$$

$$with \quad N\Delta f = 3.3$$

- For a hamming window an estimate of the required filter order is:

Example # 1 (cont.)

$$N = \frac{3.3}{\Delta f} = 330$$

- Now find the unit sample response of the ideal low pass filter that is to be windowed.
- With a cutoff frequency of $\omega_c = (\omega_s + \omega_p)/2 = 0.2\pi$ and a delay of $\alpha = N/2 = 165$
- The unit sample response is:

$$h_d(n) = \frac{\sin\left[0.2\pi(n-165)\right]}{(n-165)\pi}$$

Kaiser Windows

- Kaiser developed a family of windows that are defined by:

$$w(n) = \frac{I_0\left[\beta\left(1-\left[(n-\alpha)/\alpha\right]^2\right)^{1/2}\right]}{I_0\beta}, \quad 0 \le n \le M$$

- Where α =N/2 and I_0 is a zeroth-order modified Bessel function of the first kind, which is:

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{(x/2)^k}{k!}\right]^2$$

- Where β determines the shape of the window.
- A kaiser window is nearly optimum in the sense of having the most energy in its main lobe for a given side-lobe amplitude.

Kaiser Windows (cont.)

- Characteristics of the Kaiser Window as a function of β :

Parameter β	Side Lobe (dB)	Transition Width ($N\Delta f$)	Stopband Attenuation (dB)
2.0	-19	1.5	-29
3.0	-24	2.0	-37
4.0	-30	2.6	-45
5.0	-37	3.2	-54
6.0	-44	3.8	-63
7.0	-51	4.5	-72
8.0	-59	5.1	-81
9.0	-67	5.7	-90
10.0	-74	6.4	-99

Kaiser Windows (cont.)

- There are two empirically derived relationships for the Kaiser window that facilitate the use of these windows to design FIR filters.

- The first relates the stop band ripple of a low-pass filter α_s=-20log(δ_s) to the parameter β :

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7) & \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & 21 \le \alpha_s \le 50 \\ 0.0 & \alpha_s < 21 \end{cases}$$

- The second relates N to the transition width Δ f and the stop band attenuation α_s ,

$$N = \frac{\alpha_s - 7.95}{14.36\Delta f}, \alpha_s \ge 21$$

- Note that if α_s<21 dB a rectangular window may be used.

Example #2

- Suppose that we would like to design a low-pass filter with a cutoff frequency $\omega_c=\pi/4$, a transition width $\Delta\omega=0.02\pi$ and a stop band ripple $\delta_s=0.01$. Because $\alpha_s=-20\log(0.01)=-40$, the Kaiser window parameter is :

$$\beta = 0.5842(40-21)^{0.4} + 0.07886(40-21) = 3.4$$

- With $\Delta f = \Delta\omega / 2\pi = 0.01$ we have:

$$N = \frac{40-7.95}{14.36.(0.01)} = 224$$

- Therefore, $h(n) = h_d(n)w(n)$

$$h_d(n) = \frac{\sin[(n-112)\pi/4]}{(n-112)\pi}$$

- Is the unit sample response of the ideal low-pass filter.