-Digital Signal Processing-
FIR Filter Design-II

Lecture-18
25-May-16

Incorporation of Generalized Linear Phase

- Windows are designed with linear phase in mind
  - Symmetric around M/2

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

- So their Fourier transform are of the form

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2} \qquad \text{where } W_e(e^{j\omega}) \text{ is a real and even}$$

- Will keep symmetry properties of the desired impulse response
- Assume symmetric desired response

$$H_d(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega M/2}$$

- With symmetric window

$$A_e(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

  - Periodic convolution of real functions

Linear Phase Low pass Filter

- Desired frequency response

$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

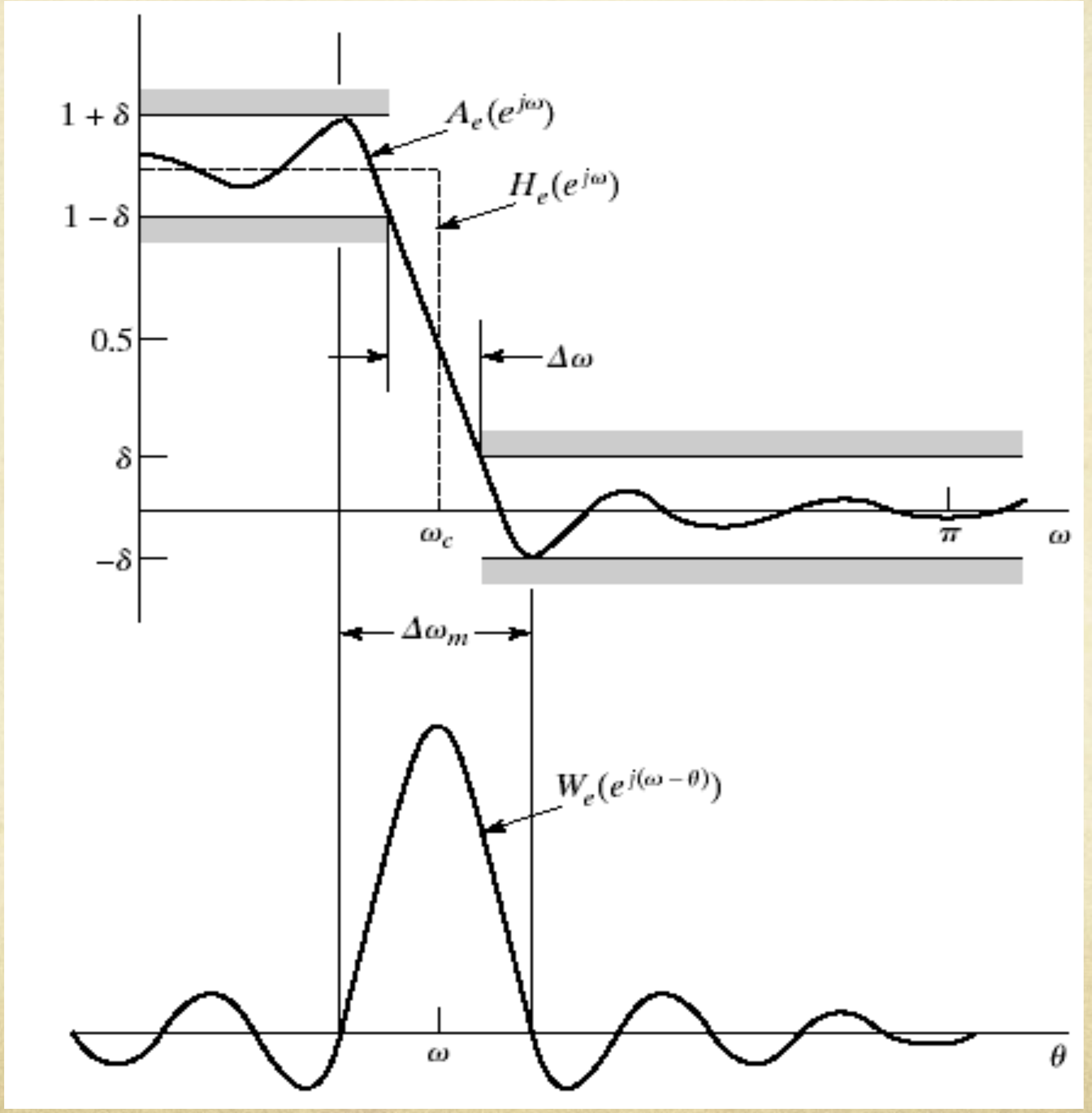
- Corresponding impulse response

$$h_{lp}[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}$$

- Desired response is even symmetric, use symmetric window

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n]$$

Linear Phase Low pass Filter (cont.)



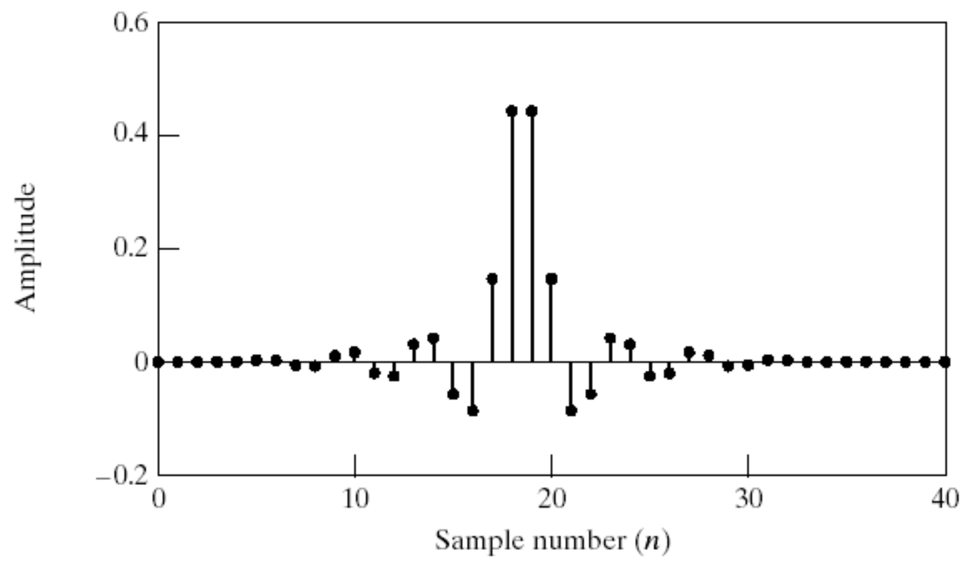
Example #1

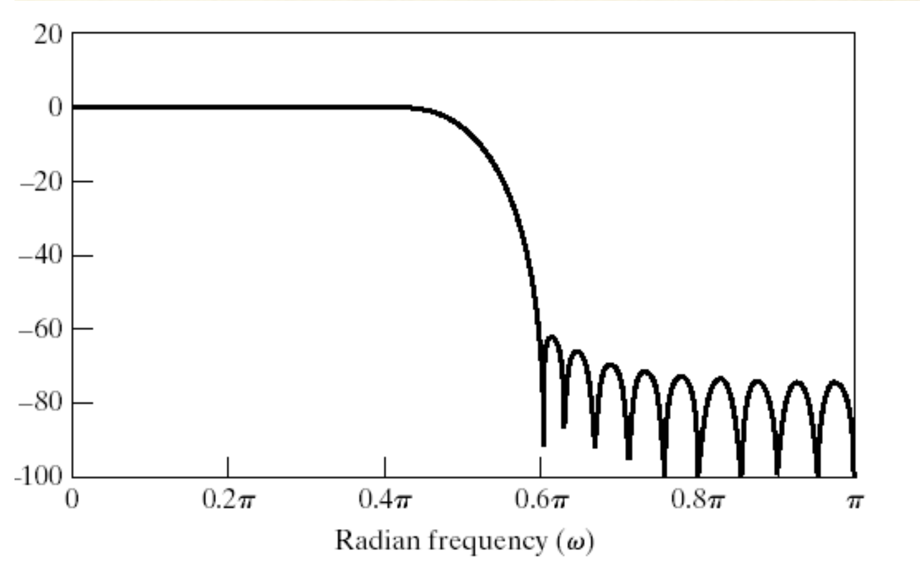
- Kaiser Window Design of a Lowpass Filter:
- Specifications $\omega_p = 0.4\pi, \omega_p = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001$
- Window design methods assume $\delta_1 = \delta_2 = 0.001$
- Determine cut-off frequency
  - Due to the symmetry we can choose it to be $\omega_c = 0.5\pi$
- Compute $\Delta\omega = \omega_s - \omega_p = 0.2\pi$ $A = -20\log_{10}\delta = 60$
- And Kaiser window parameters

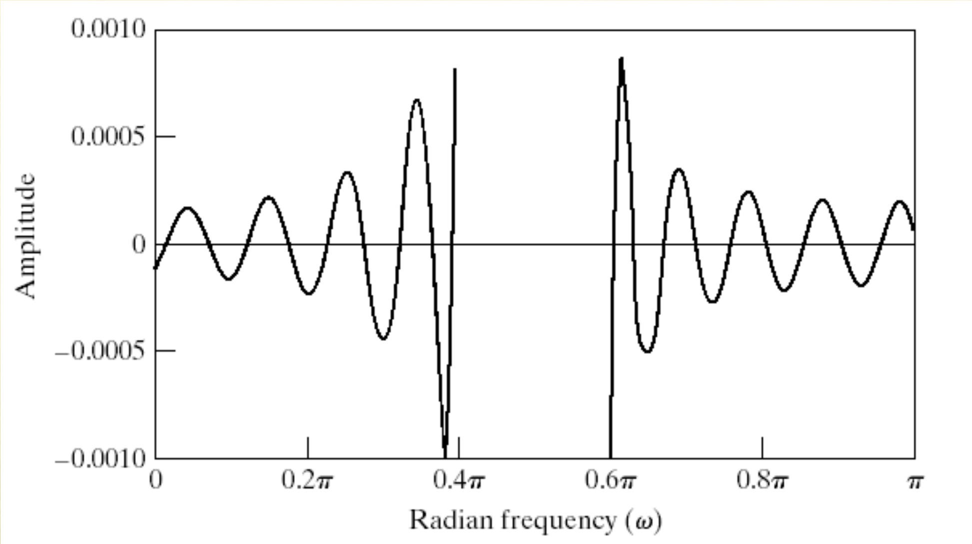
  $\beta = 5.653$ $M = 37$
- Then the impulse response is given as

$$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-18.5)]}{\pi(n-18.5)} \dfrac{I_0\left[5.653\sqrt{1-\left(\dfrac{n-18.5}{18.5}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Example #1 (cont.)





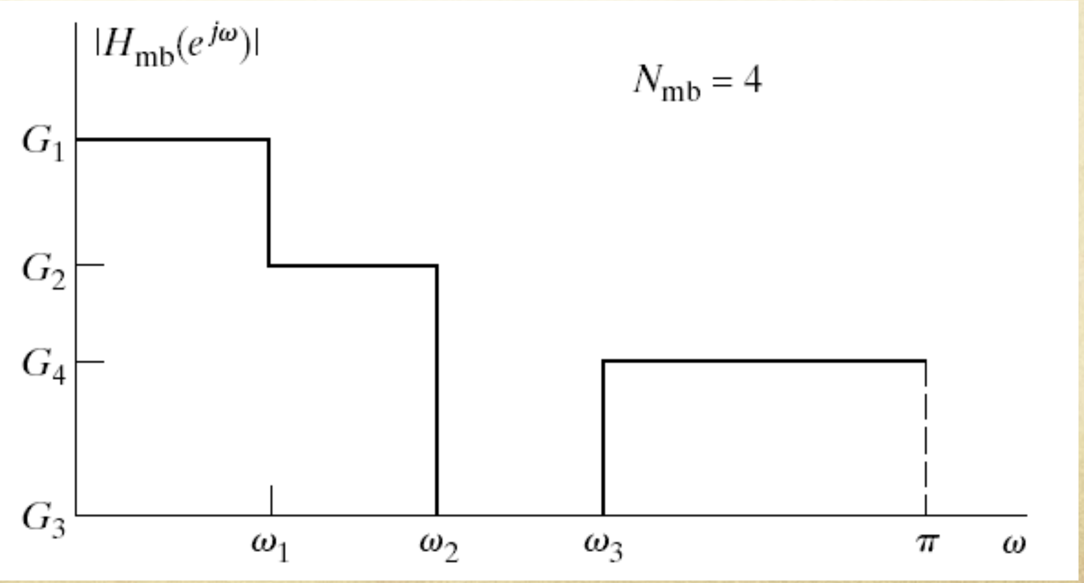


General Frequency Selective Filters

- A general multiband impulse response can be written as

$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$

- Window methods can be applied to multiband filters
- Example multiband frequency response
  - Special cases of
    - Bandpass
    - Highpass
    - Bandstop



Example #2

- We wish to use the Kaiser window method to design a discrete time filter with generalized linear phase that meets specifications of the following form:

$$\left|H\left(e^{j\omega}\right)\right| \le 0.01, \quad 0 \le |\omega| \le 0.25\pi$$

$$0.95 \le \left|H\left(e^{j\omega}\right)\right| \le 1.05, \quad 0.35\pi \le |\omega| \le 0.6\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.01, \quad 0.65 \le |\omega| \le \pi$$

- A: Determine the minimum length (M+1) of the impulse response and the value of the Kaiser window parameter β for a filter that meets the preceding specifications.
- B: What is the delay of the filter?
- C: Determine the ideal impulse response $h_d[n]$ to which the Kaiser window should be applied.

Example #2 Solution

- A: We must use the minimum specifications:

$$\delta = 0.01$$

$$\Delta\omega = 0.05\pi$$

$$A = -20\log_{10}\delta = -40dB$$

$$M + 1 = \frac{A - 8}{2.285\Delta\omega} + 1 = 90.2 \rightarrow 91$$

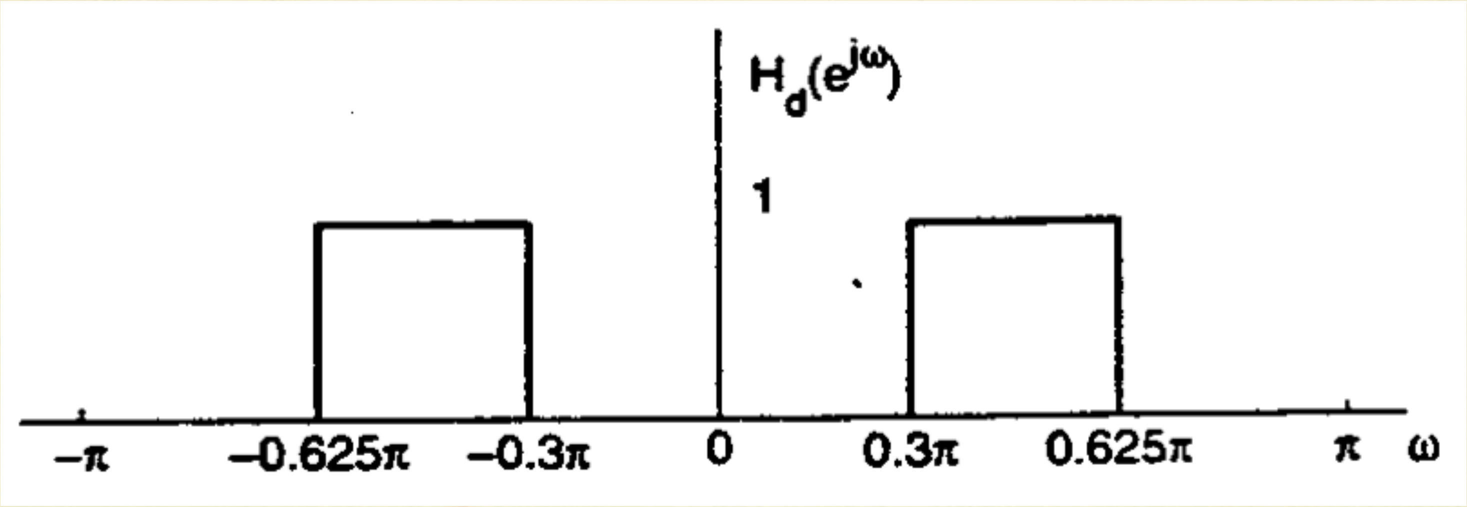
$$\beta = 0.5842\left(A - 21\right)^{0.4} + 0.07886\left(A - 21\right) = 3.395$$

- B: Since it is a linear phase filter with order 90, it has a delay of 90/2=45 samples.

- C:

$$h_d\left[n\right] = \frac{\sin\left(0.625\pi\left(n - 45\right)\right) - \sin\left(0.3\pi\left(n - 45\right)\right)}{\pi\left(n - 45\right)}$$

Example #2 Solution (cont.)



Optimum Approximations of FIR Filters

- Filter design by windows is simple but not optimal
  - Like to design filters with minimal length.
- Optimality Criterion
  - Window design with rectangular filter is optimal in one sense
    - Minimizes the mean-squared approximation error to desired response
    - But causes large error around discontinuities

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else} \end{cases} \qquad \varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right) \right|^2 d\omega$$

- Alternative criteria can give better results
  - Minimax: minimize maximize error
  - Frequency-weighted error.
- Most popular method: Parks-McClellan Algorithm
  - Reformulates filter design problem as function approximation

Optimum Approximations of FIR Filters

- In designing a causal type I linear phase FIR filter, it is convenient first to consider the design of zero-phase filter i.e.: one for which:

$$h_e[n] = h_e[-n]$$

- Insert a delay sufficient to make it causal. The corresponding frequency response is given by:

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n}$$

- With L=M/2 an integer

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n)$$

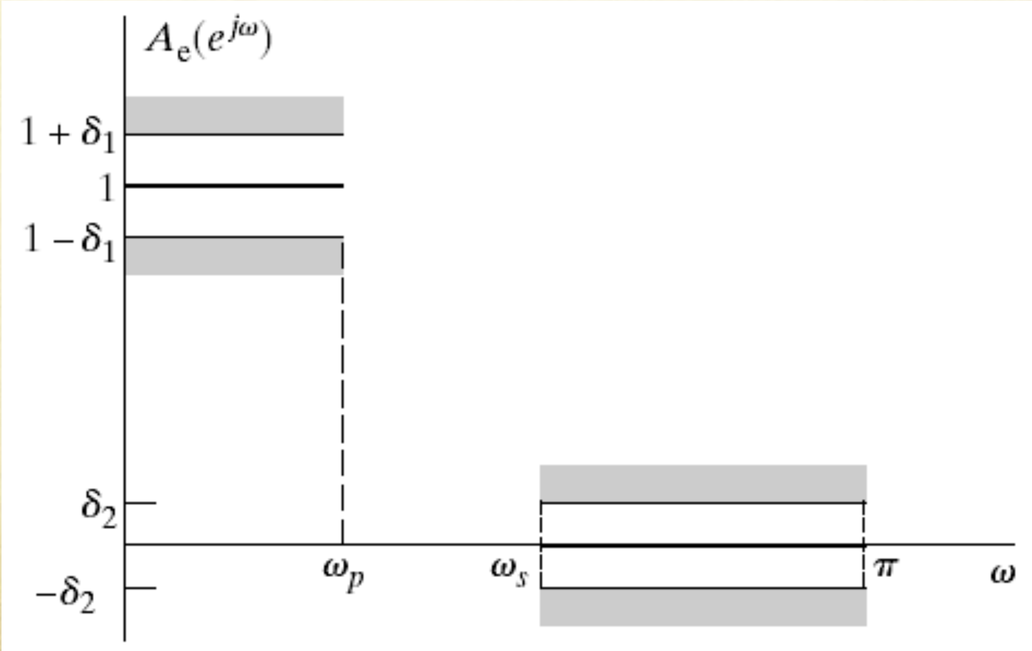
- $A_e(e^{j\omega})$ is a real, even and periodic function of ω.

Optimum Approximations of FIR Filters (cont.)

- After delaying the resulting impulse response:

$$h[n] = h_e[n - M/2] = h[M - n] \longrightarrow H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

- Goal is to approximate a desired response with $A_e(e^{j\omega})$
- Example approximation mask
  - Low-pass filter



Optimum Approximations of FIR Filters (cont.)

- Above figures shows the tolerance scheme for an approximation to a low-pass filter with a real function.
- Design algorithms have been developed in which some if the parameters L, δ_1, δ_2, ω_p, and ω_s are fixed and an iterative procedure is used to obtain optimum adjustments of the remaining parameters.
- Two distinct approaches have been developed.
- Out of which Parks-McClellan algorithm has become the dominant method for optimum design of Fir filters.
- Thus only this algorithm is

Parks-McClellan Algorithm

- It is based on reformulating the filter design problem as a problem in polynomial approximation.

- Using Chebyshev polynomials:

$$\cos(\omega n) = T_n(\cos\omega) \quad \text{where} \quad T_n(x) = \cos\left(n\cos^{-1} x\right)$$

- Where $T_n(x)$ is an nth-order polynomial.

- Express the following as a sum of powers.

$$A_e\left(e^{j\omega}\right) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- Can also be expressed as:

$$A_e\left(e^{j\omega}\right) = P(x)\Big|_{x=\cos\omega} \quad \text{where} \quad P(x) = \sum_{k=0}^{L} a_k x^k$$

Parks-McClellan Algorithm (cont.)

- Parks and McClellan showed that with L, ω_p, and ω_s fixed and convert filter design to an approximation problem.

- To formalize the approximation problem in this case, let us define an approximation error function:

$$E(\omega) = W(\omega)\left[H_d\left(e^{j\omega}\right) - A_e\left(e^{j\omega}\right)\right]$$

  - $W(\omega)$ is the weighting function
  - $H_d(e^{j\omega})$ is the desired frequency response
  - Both defined only over the passpand and stopband
  - Transition bands are unconstrained

Low-Pass Filter Approximation

- Suppose that we wish to obtain an approximation where L, ω_p, and ω_s are fixed design parameters.

$$H_d\left(e^{j\omega}\right) = \begin{cases} 1, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases}$$

- The weighing function $W(\omega)$ allows us to weigh the approximation errors differently in the different approximation intervals.

- For the low-pass filter approximation problem the weighing function is:

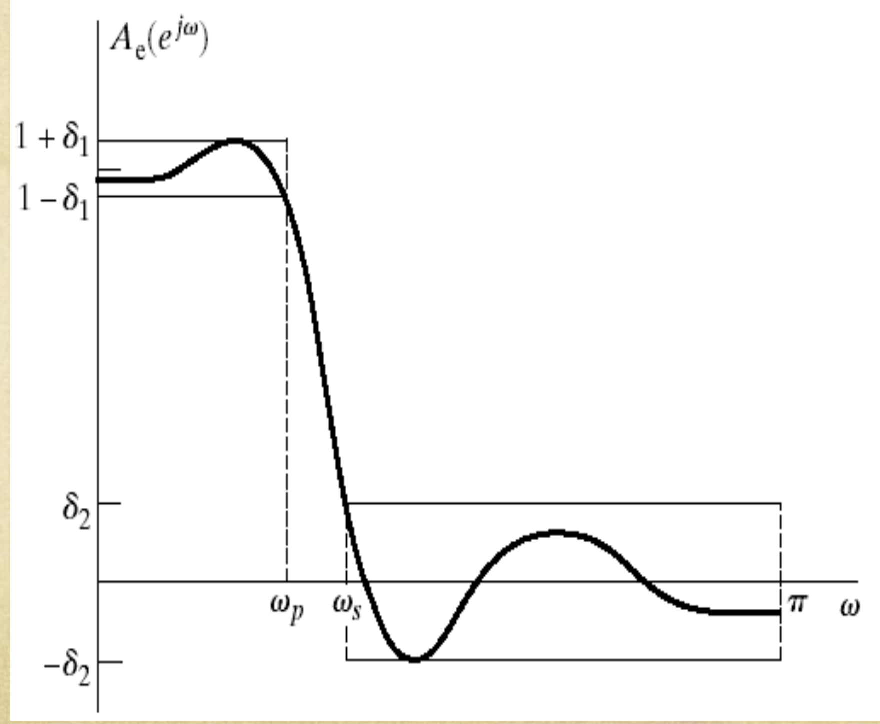
$$W(\omega) = \begin{cases} \dfrac{\delta_2}{\delta_1} & 0 \le \omega \le \omega_p \\ 1 & \omega_s \le \omega \le \pi \end{cases}$$

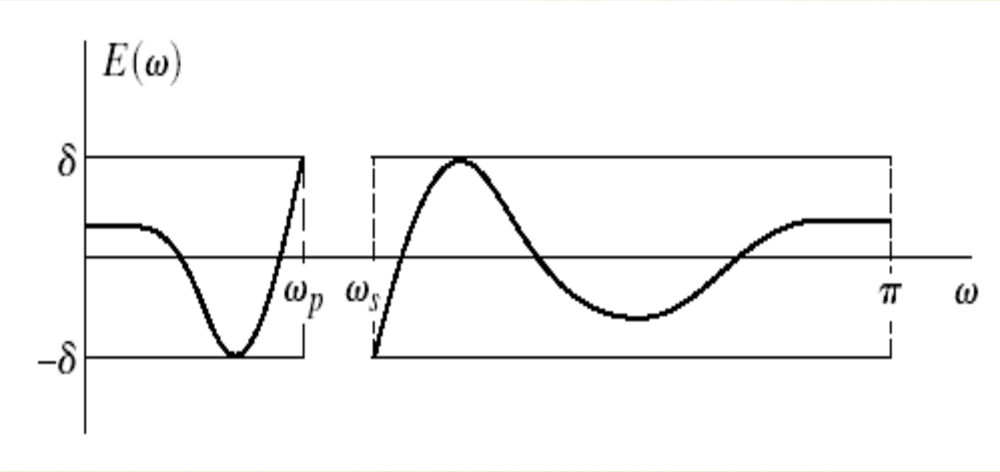
- This choice will force the error to $\delta = \delta_2$ in both bands.

- The particular criterion used in this design procedure is the so-called minimax or Chebyshev criterion.

Low-Pass Filter Approximation (cont.)

- Where within the frequency intervals of interest we seek a frequency response $A_e(e^{j\omega})$ that minimizes the maximum weighted approximation error.
- The criterion used is : $\min_{\{h_e[n]:0\leq n\leq L\}} \left(\max_{\omega\in F} |E(\omega)| \right)$
- Where F is the closed subset of $0\leq\omega\leq\pi$ such that $0\leq\omega\leq\omega_p$ or $\omega_s\leq\omega\leq\pi$.





Alternation Theorem

- Let F_p denote the closed subset consisting of the disjoint union of closed subsets of the real axis x. Then the following is the rth-order polynomial.

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

- $D_p(x)$ denotes given desired function that is continuous on F_p
- $W_p(x)$ is a positive function that is continuous on F_p
- The weighted error is given as

$$E_p(x) = W_p(x)\left[D_p(x) - P(x)\right]$$

- The maximum error is defined as

$$\|E\| = \max_{x \in F_p} \left|E_p(x)\right|$$

- A necessary and sufficient condition that P(x) be the unique r^{th} order polynomial that minimizes is that $E_p(x)$ exhibit at least (r+2) alternations.

Alternation Theorem (cont.)

- There must be at least (r+2) values x_i in F_p such that $x_1 < x_2 < \ldots < x_{r+2}$

$$E_p(x_i) = -E_p(x_{i+1}) = \mp\|E\| \quad \text{for } i = 1,2,\ldots,(r+2)$$

Example #3

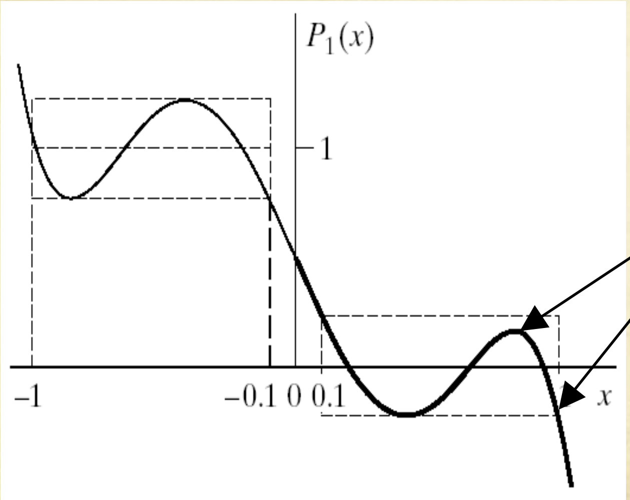
- Examine polynomials P(x) that approximate

  1 for $-1 \le x \le -0.1$

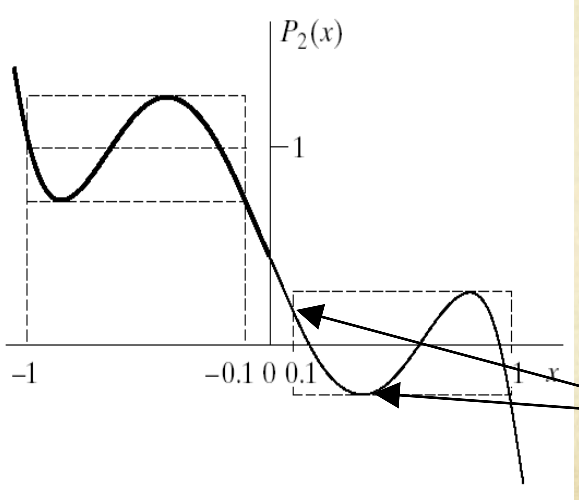
  0 for $0.1 \le x \le 1$

- Fifth order polynomials shown

- Which satisfy the theorem?

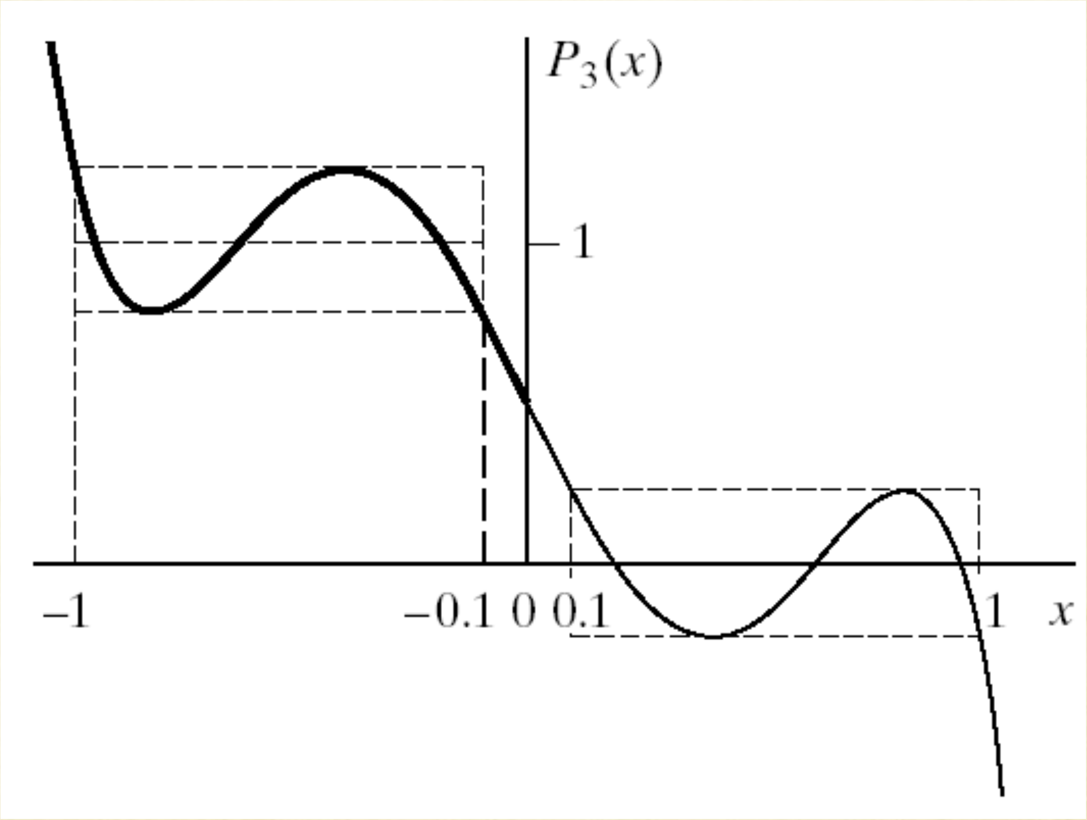


Not alternations



Not alternations

Example #3 (cont.)



Optimal Type 1 Low-Pass Filters

- In this case the P(x) polynomial is the cosine polynomial

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- The desired low-pass filter frequency response (x=cosω)

$$D_p(\cos\omega) = \begin{cases} 1 & \cos\omega_p \le \omega \le 1 \\ 0 & -1 \le \omega \le \cos\omega_s \end{cases}$$

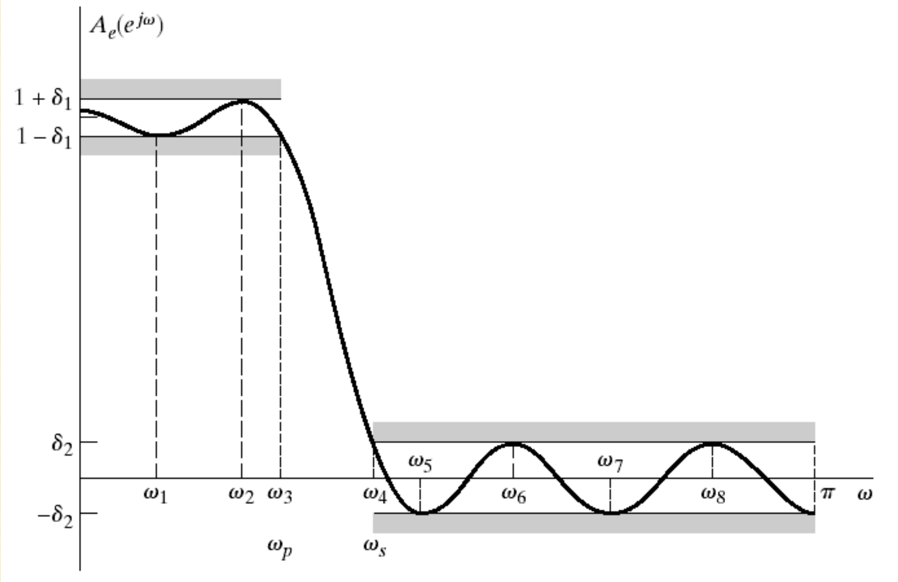
- The weighting function is given as

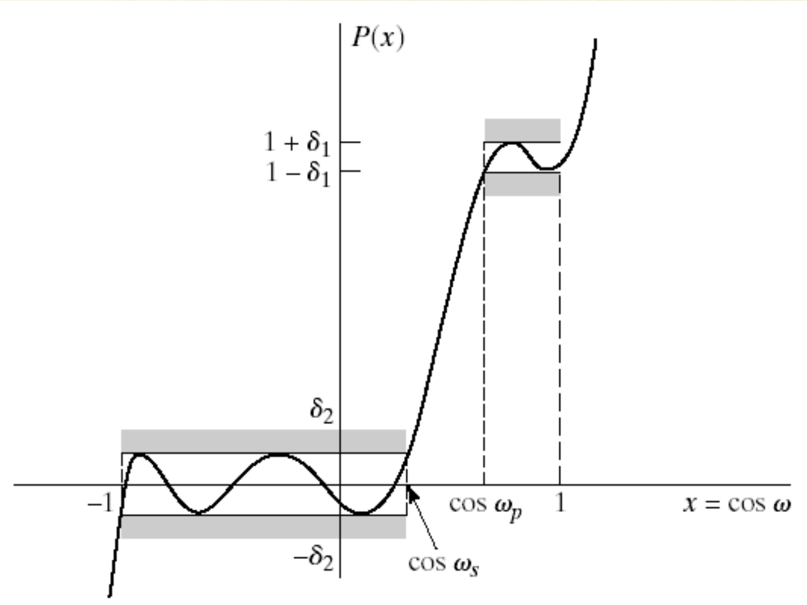
$$W_p(\cos\omega) = \begin{cases} 1/K & \cos\omega_p \le \omega \le 1 \\ 1 & -1 \le \omega \le \cos\omega_s \end{cases}$$

- The approximation error is given as

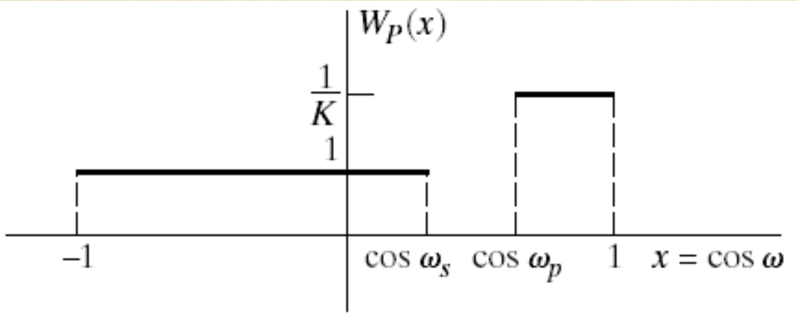
$$E_p(\cos\omega) = W_p(\cos\omega)\left[D_p(\cos\omega) - P(\cos\omega)\right]$$

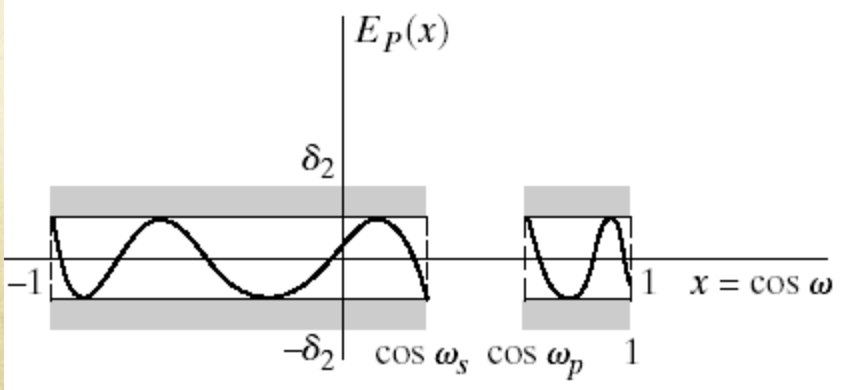
Typical Example Low-pass Filter Approximation





- 7^{th} order approximation:

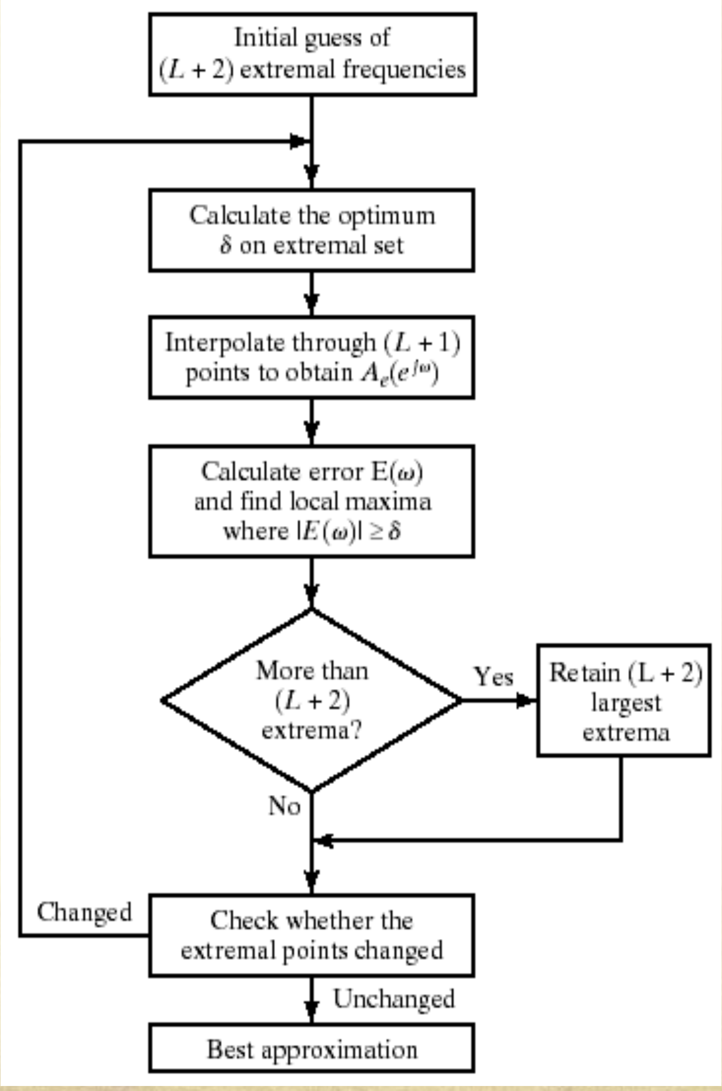




Properties of Type 1 Low-Pass Filters

- The properties are as follows:
  - The maximum possible number of alterations of the error is (L +3).
  - Alterations will always occur at ω_p and ω_s .
  - All points with zero slope inside the pass band and all points with zero slope inside the stop band will correspond to alterations, i.e., the filter will be equiripple except possibly at $\omega=0$ and $\omega=\pi$.

Flowchart of Parks-McClellan Algorithm



Example # 4

- Suppose that we would like to design an equiripple low pass filter with a pass band cutoff frequency $\omega_p = 0.3\pi$, a stop band cutoff frequency $\omega_s = 0.35\pi$, a pass band ripple of $\delta_p = 0.01$ and a stop band ripple of $\delta_s = 0.001$. estimating the filter using following equation:

$$N = \frac{-10\log\left(\delta_p - \delta_s\right) - 13}{14.6\Delta f}$$

$$N = 102$$

- Because we want the ripple in the stop band to be 10 times smaller than the ripple in the pass band, the error must be weighted using the weighting function:

$$W\left(e^{j\omega}\right) = \begin{cases} 1, & 0 \le |\omega| \le 0.3\pi \\ 10, & 0.35\pi \le |\omega| \le \pi \end{cases}$$

Example # 4 (cont.)

- Using the Parks-McClellan algorithm to design the filter, we obtain a filter with the frequency response magnitude.