# -Digital Signal Processing-
# Discrete Fourier Transform

Lecture-19
31-May-16

# Difference b/w DTFT & DFT

- Recall the DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

- The Discrete time Fourier transform is the Fourier transform of a discrete time signal. Its output is continuous in frequency and periodic.

- The Discrete Fourier transform (DFT) is the sampled version of DTFT output.

- Such a representation is very useful for digital computations and for digital hardware implementations.

# Discrete Fourier Series

- Let $\tilde{x}[n]$ be a periodic sequence with a period N:

$$\tilde{x}[n] = \tilde{x}[n+N]$$

- The Fourier series representation can be written as:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k}\tilde{X}[k]e^{j(2\pi/N)kn}$$

- Which is decomposition of $\tilde{x}[n]$ into a sum of N harmonically related complex exponentials.

- The values of Discrete Fourier series coefficients, $\tilde{X}(K)$ may be derived by multiplying both sides of this expansion by $e^{-j2\pi nl/N}$ , summing over one period and using the fact that the complex exponentials are orthogonal:

# Discrete Fourier Series

- The result is: $$\sum_{k=0}^{N-1} e^{-j2\pi n(k-l)/N} = \begin{cases} N & k = l \\ 0 & k \neq l \end{cases}$$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j2\pi nk/N}$$

- Note that the DFS coefficients are periodic with a period N:

$$\tilde{X}(k+N) = \tilde{X}(k)$$

- Hence the DFT pair is:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad \textit{Analysis}$$

$$x[n] = \frac{1}{N} \sum_{k} X[k] e^{j(2\pi/N)kn} \quad \textit{Synthesis}$$

# Example #1

- Let us find the DFT representation for the sequence:

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n-10k)$$

$$where \quad x(n) = \begin{cases} 1 & 0 \le n < 5 \\ 0 & else \end{cases}$$

- Note that $\tilde{x}[n]$ is a periodic sequence with a period N=10. Therefore, the DFS coefficients are:

$$\tilde{X}(k) = \sum_{n=0}^{9} \tilde{x}(n) e^{-j2\pi nk/10} = \sum_{n=0}^{4} e^{-j2\pi nk/10} = \frac{1-e^{-j\pi k}}{1-e^{-j\pi k/5}}$$

- Which, for 0 ≤ k ≤ 9, may be simplified to :

$$\tilde{X}(k) = \begin{cases} 5 & k=0 \\ \dfrac{2}{1-e^{-j\pi k/5}} & k \quad odd \\ 0 & k \quad even \end{cases}$$

# Example #1 (cont.)

- The DFS coefficients for all other values of k may be found from the periodicity of $\tilde{X}(K)$ i.e.,

$$\tilde{X}(k+N)=\tilde{X}(k)$$

- For convenience we sometimes use:

$$W_N = e^{-j(2\pi/N)}$$

- Analysis equation: $\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$

- Synthesis equation: $\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$

# Properties Of DFS

- Linearity:
  - The DFS pair satisfies the property of linearity. That is:

$$\tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]$$
$$\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k]$$
$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

- Shift of a Sequence:
  - If a periodic sequence $\tilde{x}[n]$ is shifted, the DFS coefficients are multiplied by a complex exponential. That is:

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]$$
$$\tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} e^{-j2\pi km/N}\tilde{X}[k]$$
$$e^{j2\pi nm/N}\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k-m]$$

# Properties Of DFS (cont.)

- Duality:
  - It is stated as: $\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k]$
    $\tilde{X}[n] \xleftrightarrow{DFS} N\tilde{x}[-k]$
- Periodic Convolution:
  - If $\tilde{h}(n)$ and $\tilde{x}(n)$ are periodic with a period N with DFS coefficients $\tilde{H}(k)$ and $\tilde{X}(k)$ , respectively the sequence with DFS coefficients:

$$\tilde{Y}(k) = \tilde{H}(k)\tilde{X}(k)$$

  - Is formed by periodically convolving $\tilde{h}(n)$ with $\tilde{x}(n)$ as follows:

$$\tilde{y}(n) = \sum_{k=0}^{N-1} \tilde{h}(k)\tilde{x}(n-k)$$

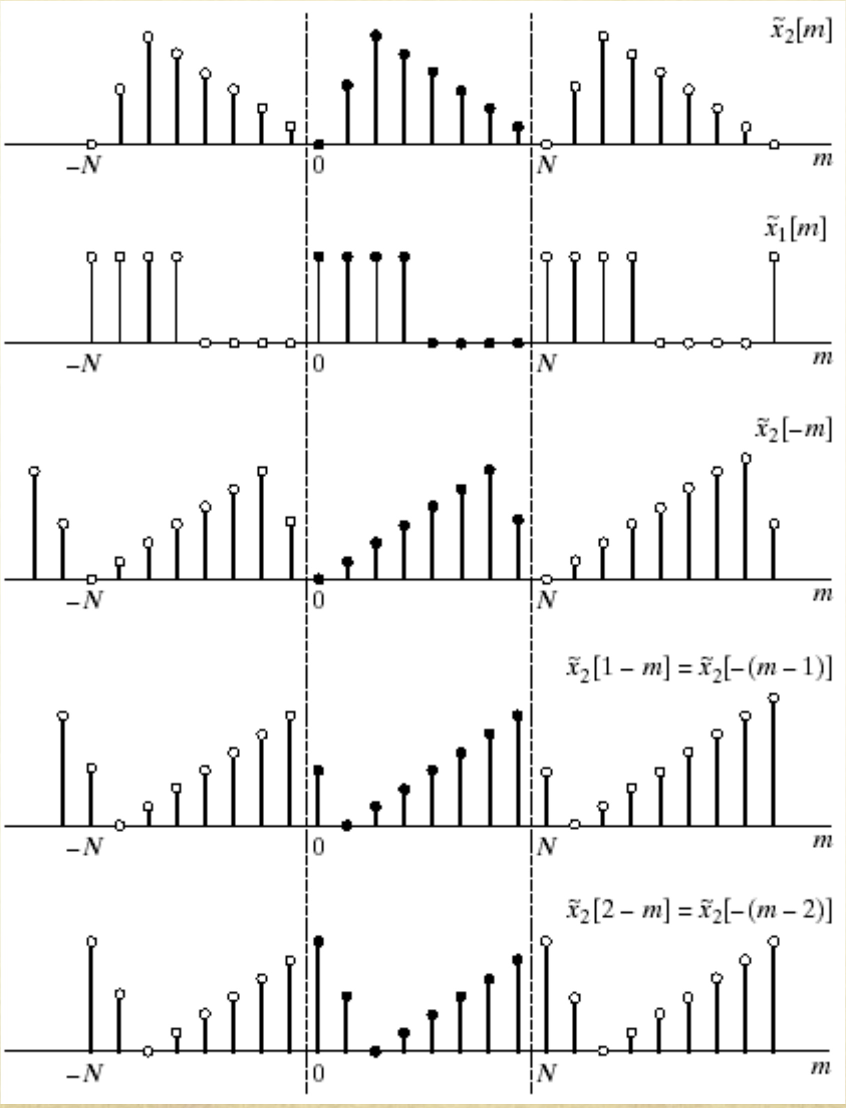
  - Notationally, the periodic convolution of two sequences is written as:

$$\tilde{y}(n) = \tilde{h}(n) * \tilde{x}(n)$$

# Properties Of DFS (cont.)

- Periodic Convolution: (cont.)
  - The only difference between periodic and linear convolution is that, with periodic convolution, the sum is only evaluated over a single period, whereas with linear convolution the sum is taken over all values of k.

# Graphical Periodic Convolution



# Symmetry Properties

| | Periodic Sequence (Period $N$) | DFS Coefficients (Period $N$) |
|---|---|---|
| 1. | $\tilde{x}[n]$ | $\tilde{X}[k]$ periodic with period $N$ |
| 2. | $\tilde{x}_1[n], \tilde{x}_2[n]$ | $\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period $N$ |
| 3. | $a\tilde{x}_1[n] + b\tilde{x}_2[n]$ | $a\tilde{X}_1[k] + b\tilde{X}_2[k]$ |
| 4. | $\tilde{X}[n]$ | $N\tilde{x}[-k]$ |
| 5. | $\tilde{x}[n-m]$ | $W_N^{km}\tilde{X}[k]$ |
| 6. | $W_N^{-\ell n}\tilde{x}[n]$ | $\tilde{X}[k-\ell]$ |
| 7. | $\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution) | $\tilde{X}_1[k]\tilde{X}_2[k]$ |
| 8. | $\tilde{x}_1[n]\tilde{x}_2[n]$ | $\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution) |
| 9. | $\tilde{x}^*[n]$ | $\tilde{X}^*[-k]$ |

# Symmetry Properties (cont.)

| Periodic Sequence (Period $N$) | DFS Coefficients (Period $N$) |
|---|---|
| 10. $\tilde{x}^*[-n]$ | $\tilde{X}^*[k]$ |
| 11. $\mathcal{R}e\{\tilde{x}[n]\}$ | $\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$ |
| 12. $j\mathcal{I}m\{\tilde{x}[n]\}$ | $\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$ |
| 13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$ | $\mathcal{R}e\{\tilde{X}[k]\}$ |
| 14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$ | $j\mathcal{I}m\{\tilde{X}[k]\}$ |
| Properties 15–17 apply only when $x[n]$ is real. | |
| 15. Symmetry properties for $\tilde{x}[n]$ real. | $\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$ |
| 16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$ | $\mathcal{R}e\{\tilde{X}[k]\}$ |
| 17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$ | $j\mathcal{I}m\{\tilde{X}[k]\}$ |

# Discrete Fourier Transform

- The DFT is an important decomposition for sequences that are finite in length.
- Whereas the DTFT is a mapping from a sequence to a function of a continuous variable $\omega$,

$$x(n) \overset{DTFT}{\Leftrightarrow} X\left(e^{j\omega}\right)$$

- The DFT is a mapping from a sequence, x(n) to another sequence X(k),

$$x(n) \overset{DFT}{\Leftrightarrow} X(k)$$

- The DFT may be easily developed from the Discrete Fourier series representation for periodic sequences.
- The DFT pair was given as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

# DFT Properties

- Linearity:
  - If $x_1(n)$ and $x_2(n)$ have N-point DFTs $X_1(k)$ and $X_2(k)$ , respectively:

$$ax_1(n)+bx_2(n)\overset{DFT}{\Leftrightarrow} aX_1(k)+bX_2(k)$$

  - In order to use this property it is important to make sure that the DFTs are the same length.

- Symmetry:
  - If x(n) is real valued, X(k) is conjugate symmetric:

$$X(k)=X^*((-K))=X^*((N-K))_N$$

  - If x(n) is imaginary, X(k) is conjugate anti-symmetric:

$$X(k)=-X^*((-K))=-X^*((N-K))_N$$

# DFT Properties (cont.)

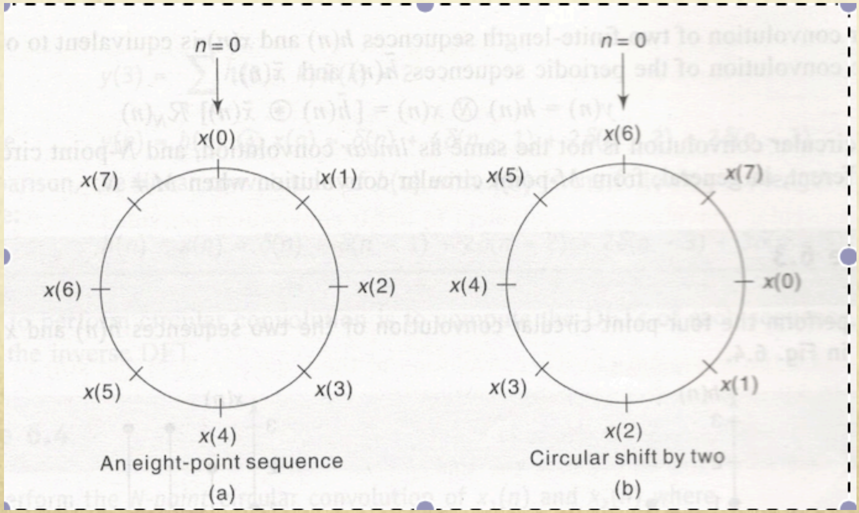
- Circular Shift:
  - The circular shift of a sequence x(n) is defined as follows:

$$X\left(\left(n-n_0\right)\right)_N R_N\left(n\right)=\tilde{x}\left(n-n_0\right)R_N\left(n\right)$$

  - Where $n_0$ is the amount of shift ad $R_N(n)$ is a rectangular window:

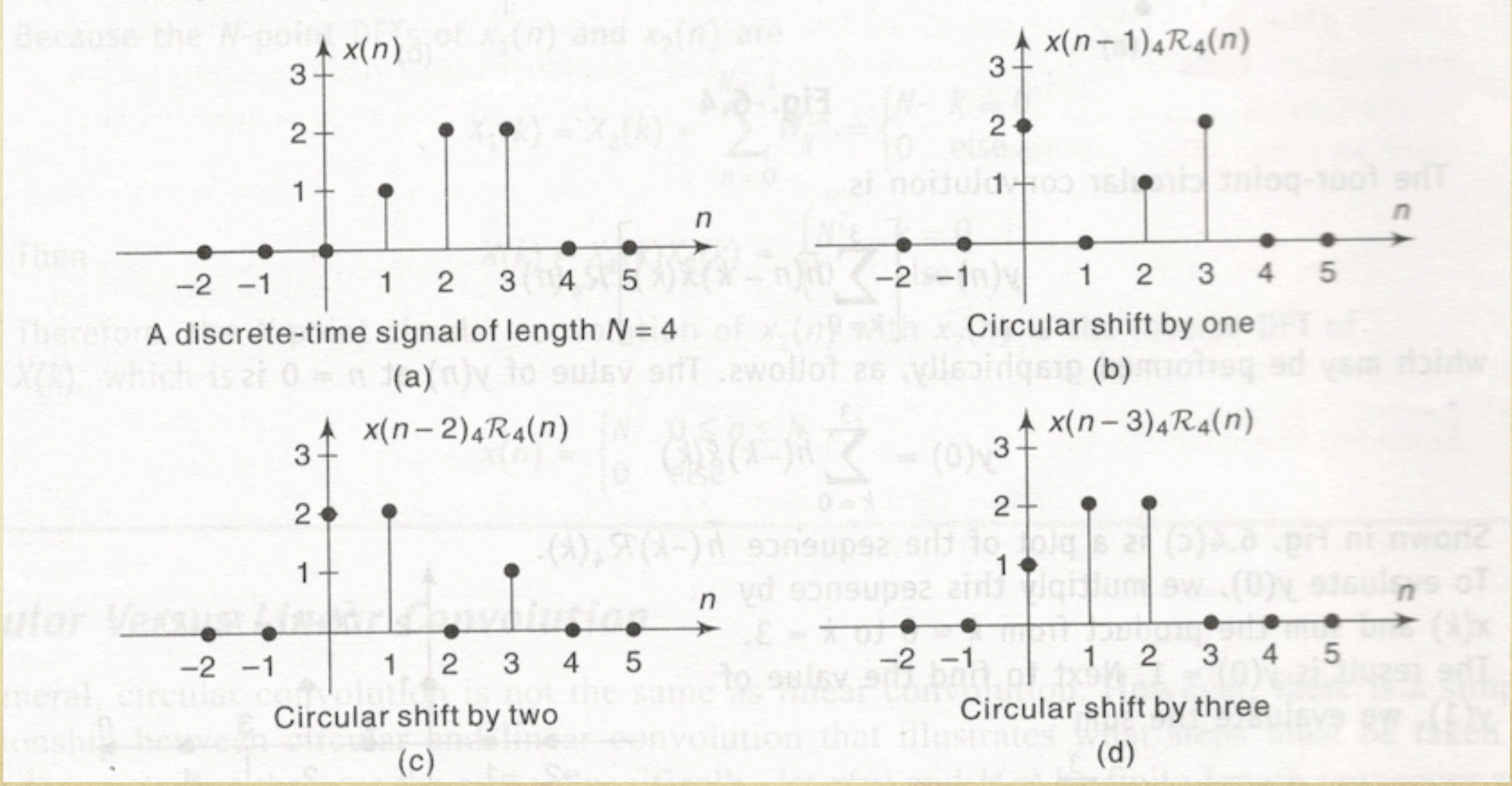
$$R_N\left(n\right)=\begin{cases}1 & 0\le n<N\\ 0 & else\end{cases}$$

  - A circular shift may be visualized as follows:



# DFT Properties (cont.)

- Circular Shift: (cont.)



# DFT Properties (cont.)

- Circular Shift: (cont.)
  - If a sequence is circularly shifted, the DFT is multiplied by a complex exponential: $x(n-n_0)R_N(n) \overset{DFT}{\Leftrightarrow} W_N^{n_0 k} \quad X(k)$
  - Similarly with a circular shift of the DFT, $X((k\text{-}k_0))_N$, the sequence is multiplied by a complex exponential:

$$W_N^{nk_0} \quad x(n) \quad X\left(\left(k+k_0\right)\right)_N$$

- Circular Convolution:
  - Let h(n) and x(n) be finite-length sequences of length N with N-point DFTs H(k) and X(k), respectively.
  - The sequence that has a DFT equal to the product Y(k)=H(k)X(k) is:

$$y(n)=\left[\sum_{k=0}^{N-1}\tilde{h}(k)\tilde{x}(n-k)\right]R_N(n)=\left[\sum_{k=0}^{N-1}\tilde{h}(n-k)\tilde{x}(k)\right]R_N(n)$$

# DFT Properties (cont.)

- Circular Convolution: (cont.)
  - Where $\tilde{x}(n)$ and $\tilde{h}(n)$ are the periodic extensions of the sequences x(n) and h(n), respectively.
  - Because $\tilde{h}(n) = h(n)$ for 0≤n<N. The previous equation can also be written as:

$$y(n) = \left[\sum_{k=0}^{N-1} h(k)\tilde{x}(n-k)\right] R_N(n)$$

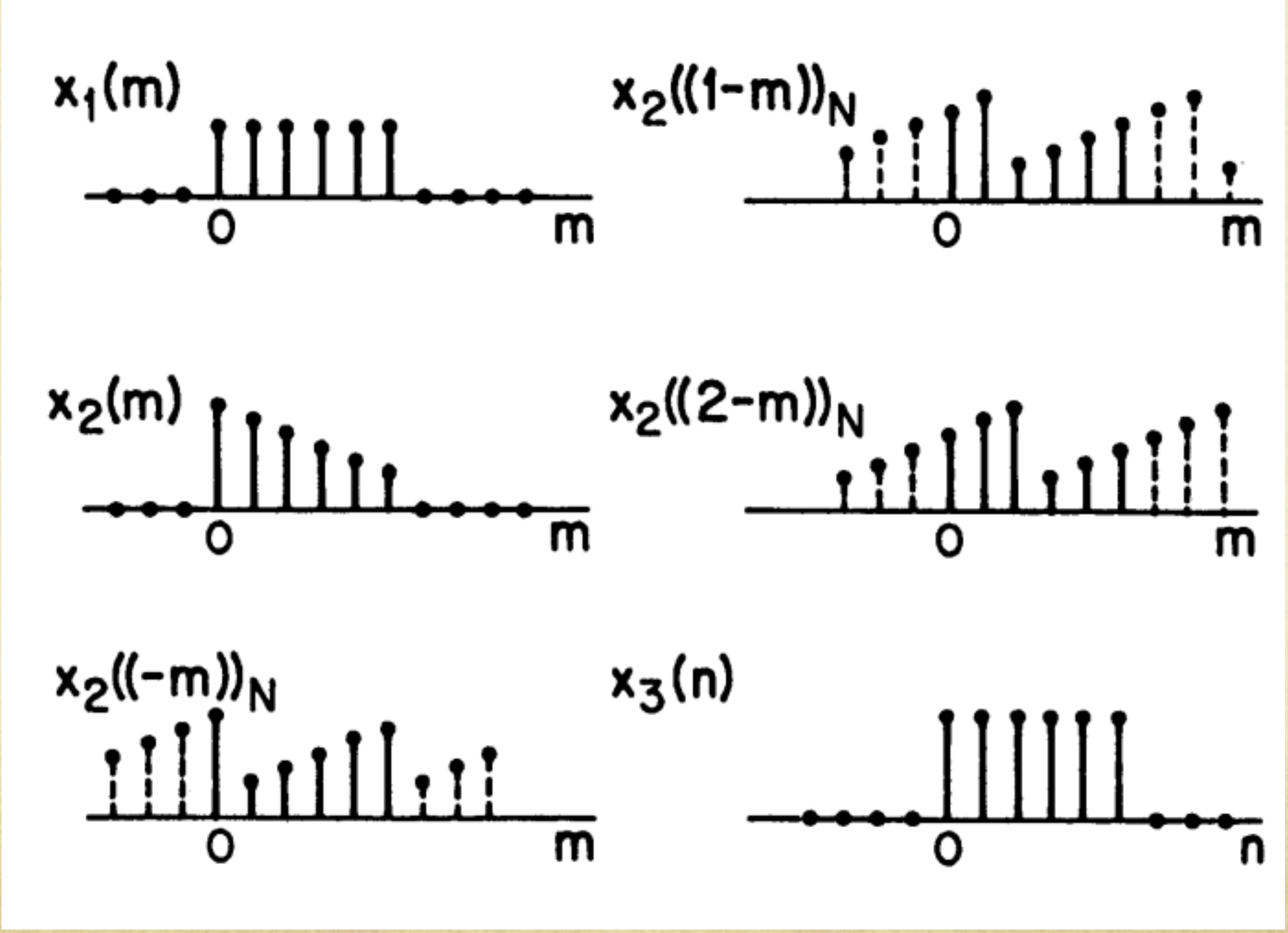
  - The sequence y(n) in above equation in the N-point circular convolution of h(n) with x(n) and it is written as:

$$y(n) = h(n) \circledN x(n) = x(n) \circledN h(n)$$

  - The circular convolution of two finite-length sequences h(n) and x(n) is equivalent to one period of the periodic convolution of the periodic sequences $\tilde{h}(n)$ and $\tilde{x}(n)$ , (equation scan from book)

# Example #2

- Example of circular convolution of two sequences:



# Linear Convolution Using DFT

- The DFT provides a convenient way to perform convolutions without having to evaluate the convolution sum.
- If h(n) is $N_1$ points long and x(n) is $N_2$ points long, h(n) may be linearly convolved with x(n) as follows:
  - Pad the sequences h(n) and x(n) with zeros so that they are of length $N \geq N_1 + N_2 - 1$.
  - Find the N-point DFTs of h(n) and x(n).
  - Multiply the DFTs to form the product Y(k) = H(k) X(k).
  - Find the inverse DFT of Y(k).
- In spite of its computational advantages there are some difficulties with the DFT approach.
- For example, if x(n) is very long we must commit a significant amount of time computing very long DFTs and in the process accept very long processing delays.
- The solution to this problem is to use block convolution, which involves segmenting the signal to be filters x(n) into sections.

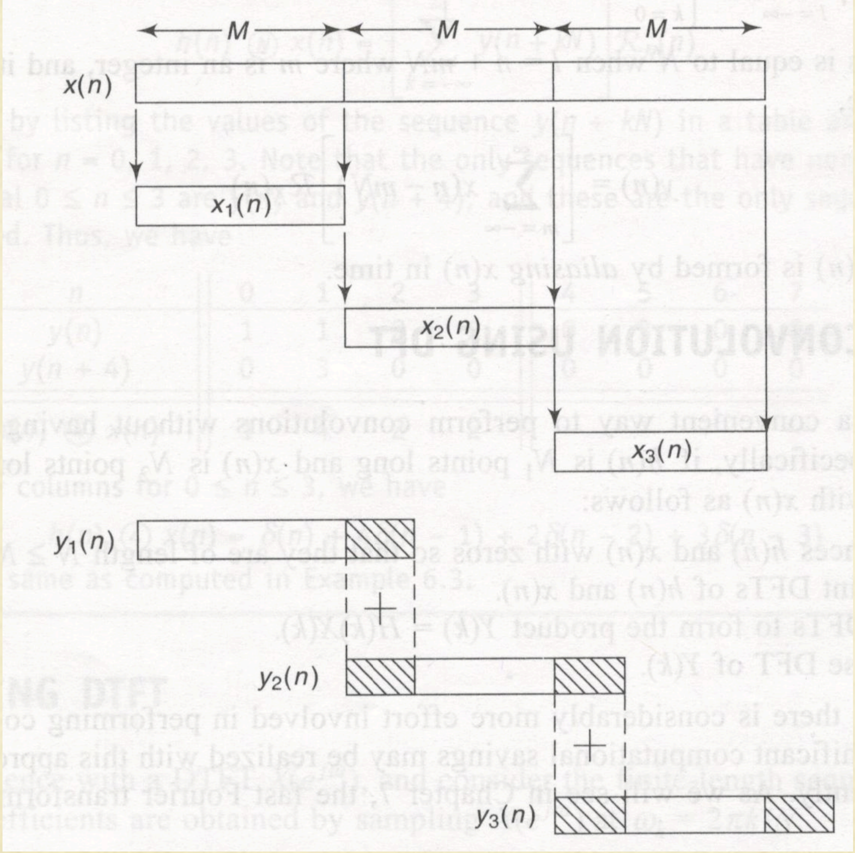
# Linear Convolution Using DFT (cont.)

- Each section is then filtered with the Fir filter h(n), and the filtered sections are pieced together to form the sequence y(n).
- There are two block convolution techniques:
  - Overlap-add
  - Overlap-save

# Overlap-add

- Let x(n) be a sequence that is to be convolved with a causal FIR filter h(n) of length L: $y(n) = h(n) * x(n) = \sum_{k=0}^{L-1} h(k)x(n-k)$
- Assume that x(n)=0 for n<0 and that the length of x(n) is much greater than L.
- In this method x(n) is partitioned into non-overlapping subsequences of length M as shown below:

# Overlap-add (cont.)



# Overlap-add (cont.)

- Thus x(n) may be written as a sum of shifted finite-length sequences of length M:

$$x(n) = \sum_{i=0}^{\infty} x_i(n - Mi)$$

$$where \quad x_i = \begin{cases} x(n + Mi) & n = 0,1,\ldots,M-1 \\ 0 & else \end{cases}$$

- Therefore the linear convolution of x(n) with h(n) is:

$$y(n) = h(n) * x(n) = \sum_{i=0}^{\infty} x_i(n - Mi) * h(n) = \sum_{i=0}^{\infty} y_i(n - Mi)$$

- Where $y_i(n)$ is the linear convolution of $x_i(n)$ with h(n):
  $y_i(n) = x_i(n)*h(n)$

- Each sequence $y_i(n)$ is of length N=L+M-1, it may be found by multiplying the N-point DFTs of $x_i(n)$ and h(n)

# Overlap-add (cont.)

- The reason for the name overlap-add is that for each I, the sequences $y_i(n)$ and $y_{i+1}(n)$ overlap at (N-M) points and in performing the sum these overlapping points are added.

# Overlap-save

- This method takes advantage of the fact that the aliasing that occurs in circular convolution only affects a portion of the sequences.

- For example if x(n) and h(n) are finite length sequences of lengths L and n respectively, the linear convolution y(n) is a finite length sequences of lengths N+L-1.

- Therefore assuming that N>L, if we perform an N-point circular convolution of x(n) with h(n):

$$h(n) \circledN x(n) = \left[ \sum_{k=-\infty}^{\infty} y(n+kN) \right] \mathcal{R}_N(n)$$

- Because y(n+N) is the only term that is aliased into the interval $0 \le n \le N-1$ , and because y(n+N) only overlaps the first L-1 values of y(n) and the remaining values in the circular convolution will not be aliased.

- In other words the first L-1 values of the circular convolution are not equal to the linear convolution, whereas the last M=N-L+1 values are the same. (shown in the figure below)

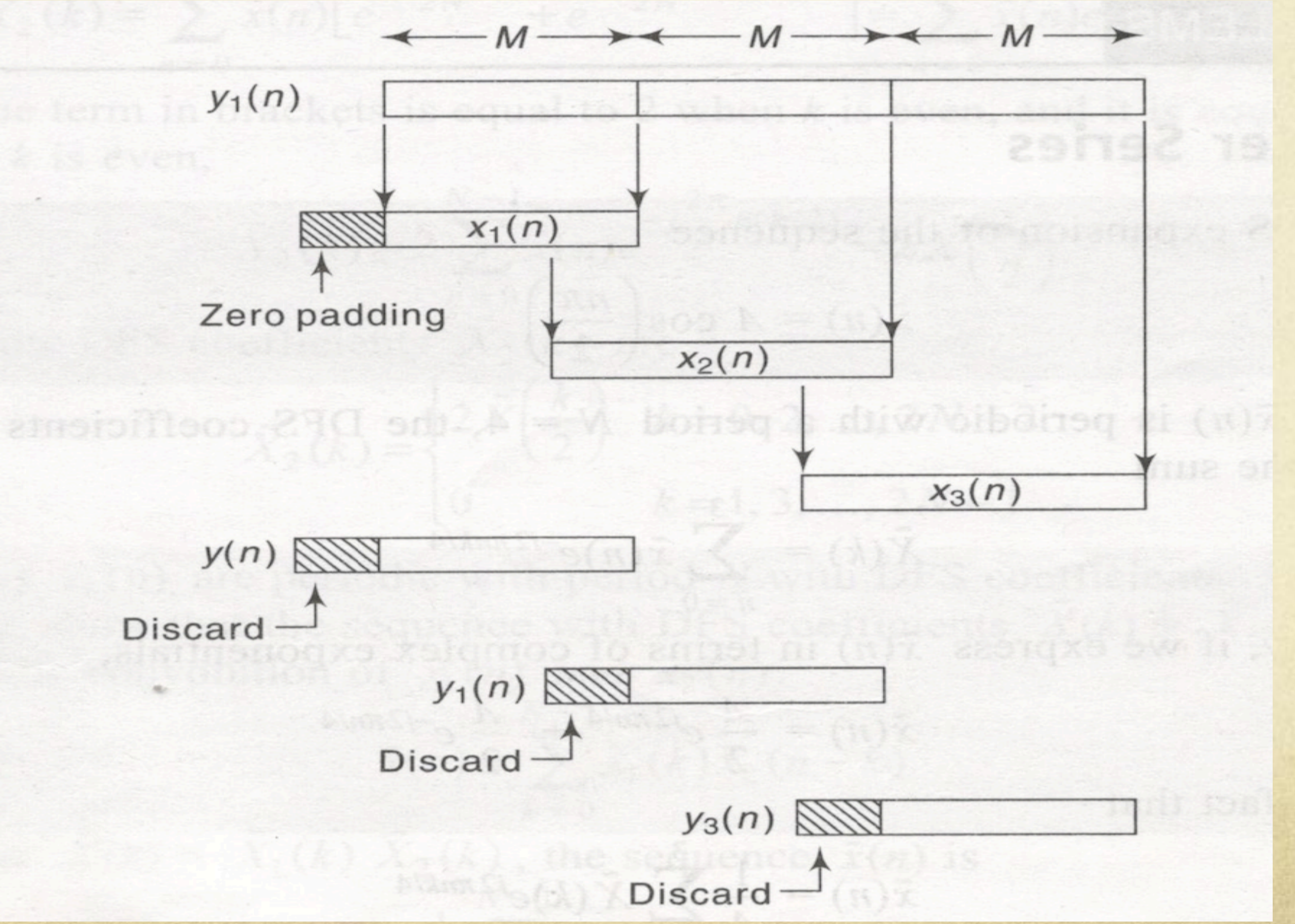
- Thus with the appropriate partitioning of the input sequence x(n), linear convolution may be performed by piecing together circular convolutions.

# Overlap-save (cont.)

- The procedure is as follows:
  - Let $x_1(n)$ be the sequence: $$x_1(n) = \begin{cases} 0 & 0 \le n < L-1 \\ x(n-L+1) & L-1 \le n \le N-1 \end{cases}$$
  - Perform the N-point circular convolution of $x_1(n)$ with h(n) by forming the product $H(k)X_1(k)$ and then finding the inverse DFT, $y_1(n)$. The first L-1 values of the circular convolution are aliased and the last N-L+1 values corresponds to the linear convolution of x(n) with h(n). Due to the zero padding at the start of $x_1(n)$, these last N-L+1 values are the first N-L+1 values of y(n): $$y(n) = y_1(n+L-1), \quad 0 \le n \le N-L$$
  - Let $x_2(n)$ be the N-point sequence that is extracted from x(n) with the first L-1 values overlapping with those of $x_1(n)$.
  - Perform N-point circular convolution of $x_2(n)$ with h(n)) by forming the product $H(k)X_2(k)$ and taking the inverse DFT. The first L-1 values of $y_2(n)$ are discarded and the final N-L+1 values are saved and concatenated with the saved values of $y_1(n)$: $$y(n+N-L+1) = y_2(n+L-1), \quad 0 \le n \le N-L$$

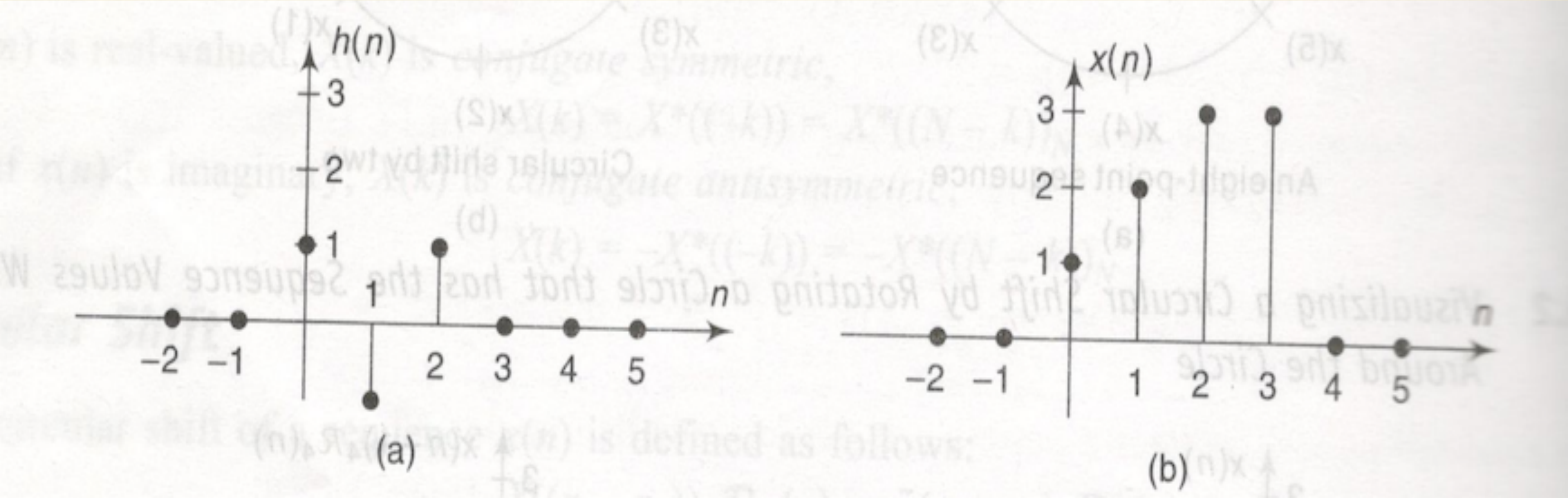
# Overlap-save (cont.)

- Steps 3 and 4 are repeated until all of the values in the linear convolution have been evaluated.



# Example #3

- Let us perform the four-point circular convolution of the two sequences h(n) and x(n), shown below:



# Example #4

- Suppose we have two four-point sequences x[n] and h[n] as follows:

$$x[n] = \cos\left(\frac{\pi n}{2}\right), \quad n = 0,1,2,3$$

$$h[n] = 2^n, \quad n = 0,1,2,3$$

  - (a): Calculate the four-point DFT X[k].
  - (b): Calculate the four-point DFT H[k].
  - (c): Calculate $y[n] = x[n] \circledast_4 h[n]$ by doing the circular convolution directly.
  - (d): Calculate y[n] of Part (c) by multiplying the DFTs of x[n] and h[n] and performing an inverse DFT.