Solutions to the Examples in Lecture#20:

Solution of Example #1:

8.1. We sample a periodic continuous-time signal with a sampling rate:

$$F_s = \frac{\Omega s}{2\pi} = \frac{1}{T} = \frac{6}{10^{-3}} \text{Hz}$$

(a) The sampled signal is given by:

$$x[n] = x_c(nT)$$

Expressed as a Discrete Fourier Series:

$$x[n] = \sum_{k=-9}^{9} a_k e^{j\frac{2\pi}{6}kn}$$

We note that, in accordance with the discussion of Section 8.1, the sampled signal is represented by the summation of harmonically-related complex exponentials. The fundamental frequency of this set of exponentials is $2\pi/N$, where $N = 6$.

Therefore, the sequence $x[n]$ is periodic with period 6.

(b) For any bandlimited continuous-time signal, the Nyquist Criterion may be stated from Eq. (4.14b) as:

$$F_s \geq 2F_N,$$

where F_s is the sampling rate (Hz), and F_N corresponds to the highest frequency component in the signal (also Hz).

As evident by the finite Fourier series representation of $x_c(t)$, this continuous-time signal is, indeed, bandlimited with a maximum frequency of $F_n = \frac{9}{10^{-3}}$ Hz.

Therfore, by sampling at a rate of $F_s = \frac{6}{10^{-3}}$ Hz, the Nyquist Criterion is violated, and aliasing results.

Solution of Example #2:

8.7. We have a six-point uniform sequence, $x[n]$, which is nonzero for $0 \le n \le 5$. We sample the Z-transform of $x[n]$ at four equally-spaced points on the unit circle.

$$X[k] = X(z)|_{z=e^{(2\pi k/4)}}$$

We seek the sequence $x_1[n]$ which is the inverse DFT of $X[k]$. Recall the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is zero for all n outside $0 \le n \le 5$, we may replace the infinite summation with a finite summation. Furthermore, after substituting $z = e^{j(2\pi k/4)}$, we obtain

$$X[k] = \sum_{n=0}^{5} x[n]W_4^{kn}, \quad 0 \le k \le 4$$

Note that we have taken a 4-point DFT, as specified by the sampling of the Z-transform; however, the original sequence was of length 6. As a result, we can expect some aliasing when we return to the time domain via the inverse DFT.

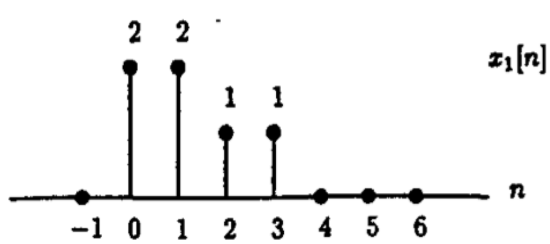
Performing the DFT,

$$X[k] = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}, \quad 0 \le k \le 4$$

Taking the inverse DFT by inspection, we note that there are six impulses (one for each value of n above). However,

$$W_4^{4k} = W_4^{0k} \text{ and } W_4^{5k} = W_4^{k},$$

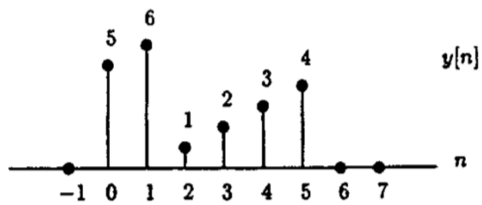
so two points are aliased. The resulting time-domain signal is



Solution of Example #3

8.11. We wish to perform the circular convolution between two 6-pt sequences. Since $x_2[n]$ is just a shifted impulse, the circular-convolution coincides with a circular shift of $x_1[n]$ by two points.

$$
\begin{aligned}
y[n] &= x_1[n] \circledS x_2[n] \\
&= x_1[n] \circledS \delta[n-2] \\
&= x_1[((n-2))_6]
\end{aligned}
$$



8.12 (a)

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Solution of Example #4:

(a)

$$
\begin{aligned}
x[n] &= \delta[n] \\
X[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
x[n] &= \delta[n-n_0], \quad 0 \le n_0 \le (N-1) \\
X[k] &= \sum_{n=0}^{N-1} \delta[n-n_0] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= W_N^{kn_0}
\end{aligned}
$$

(e)

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq (N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} a^n W_N^{kn}, \quad 0 \leq k \leq (N-1)$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j(2\pi k)/N}}$$

$$X[k] = \frac{1 - a^N}{1 - a e^{-j(2\pi k)/N}}$$

Solution of Example #5:

(a)

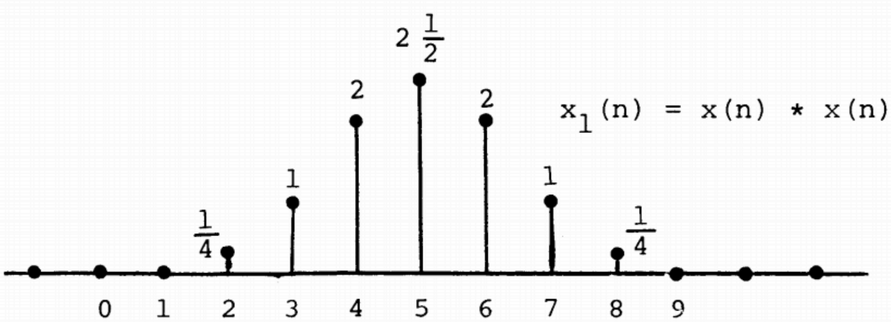
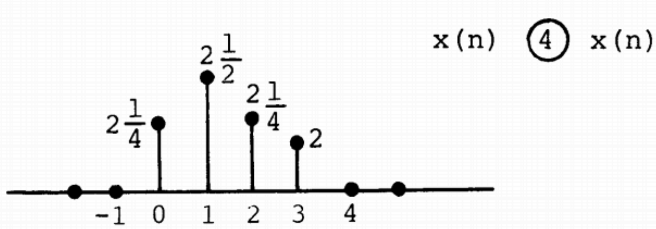


Figure S10.2-1

(b) We can obtain the four-point circular convolution by "aliasing" the linear convolution. Thus

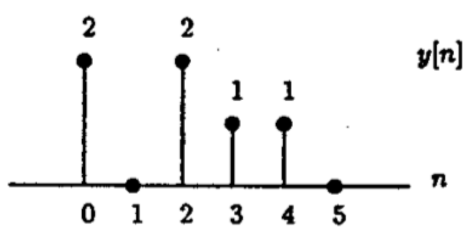
$$x(n) \; ④ \; x(n) = \left[\sum_{r=-\infty}^{+\infty} x_1(n + 4r)\right] R_4(n)$$



(c) The ten-point circular convolution can be obtained in the same way. In this case, however, since $x_1(n)$ is of length 7, the delayed replicas of $x_1(n)$ in the "aliasing" equation do not overlap. Thus $x(n)$ ⑩ $x(n)$ is identical to the linear convolution as obtained in (a).

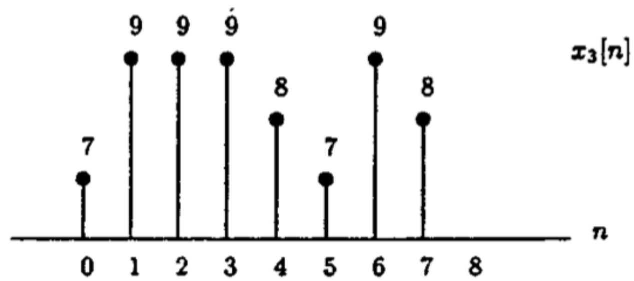
Solution of Example #6:

8.13. Using the properties of the DFT, we get $y[n] = x[((n-2))_5]$, that is $y[n]$ is equal to $x[n]$ circularly shifted by 2. We get:



Solution of Example #7:

8.14. $x_3[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 8$. Carrying out the 8-point circular convolution, we get:



We thus conclude $x_3[2] = 9$.