Linear Algebra

System of Linear Equations & Matrices

14th July 16

Course Assessment

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Course Assessment

- Total Assessment 100%.
  - Final Exam : 80%
  - Internal Evaluation : 20%

- Internal Evaluation 20%
  - Quizzes : 10%
    - Total Quizzes 6: Best of 5
  - Assignment : 10%
    - Total Assignments 6: Best of 5

Introduction

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Matrices

- Information in science, business and mathematics is often organized into rows and columns to form rectangular arrays called "Matrices"
- Tables of numerical data that arise from physical observations
- For example: the information required to solve a system of equations such as:

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

- Solution is obtained by preforming appropriate operations on this matrix.

System of Linear Equations

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Linear Equations

- In a rectangular xy-coordinate system can be represented by an equation of the form: $ax + by = c$
  - Where a, b and c are real constants.
- In n variables: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$
  - Where $a_1,\ldots\ldots,a_n$ and b are real constants.
  - $x_1,\ldots\ldots x_n$ = unknowns.
- Example 1:
  - The linear equations does not involve any products or roots of variables.
  - The following are linear equations:

$$x + 3y = 7 \qquad y = \tfrac{1}{2}x + 3z + 1 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 7$$

  - The following equations are not linear:

$$x + \sqrt[3]{y} = 5 \qquad 3x + 2y - z + xz = 4 \qquad y = \sin x$$

Linear Equations

- The finite set of linear equation is called a system of linear equations or linear system.
- The variables are known as unknowns.
- Example 2 : Finding a Solution Set:

$$4x - 2y = 1$$

  - 1 equation and two unknowns. We will set one variable as the parameter.
  - $x = t, y = 2t - \frac{1}{2}$ or $x = \frac{1}{2}t + \frac{1}{4}, y = t$

$$x_1 - 4x_2 + 7x_3 = 5$$

  - 1 equation and 3 unknowns. We will set 2 variables as parameter.

$$x_1 = 5 + 4s - 7t, x_2 = s, x_3 = t$$

Linear Systems

- Linear systems in two unknowns arise in connection with intersections of lines.
- Linear system is consistent if it has at least one solution and inconsistent if it has no solutions.
- Thus a consistent linear system of two equations in two unknowns has either one solution or infinitely many solutions.
- The same is true for a linear system of three equations in three unknowns.

Example #1

- A linear system with one solution:
- Solve the linear system:

$$x - y = 1$$
$$2x + y = 6$$

- System has the unique solution:

$$x = \frac{7}{3}, y = \frac{4}{3}$$

Example #2

- A linear system with Infinitely many solutions:
- Solve the linear system:

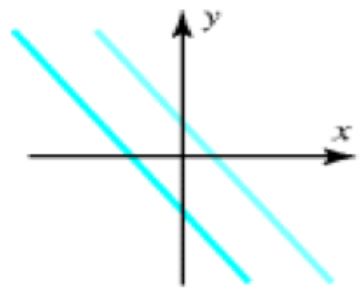
$$x - y + 2z = 5$$

$$2x - 2y + 4z = 10$$

$$3x - 3y + 6z = 15$$

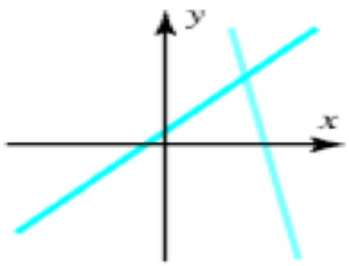
Linear Systems

- (x,y) lies on a line if and only if the numbers x and y satisfy the equation of the line. Solution: points of intersection l1 & l2
  - l1 and l2 may be parallel: no intersection, no solution



No solution

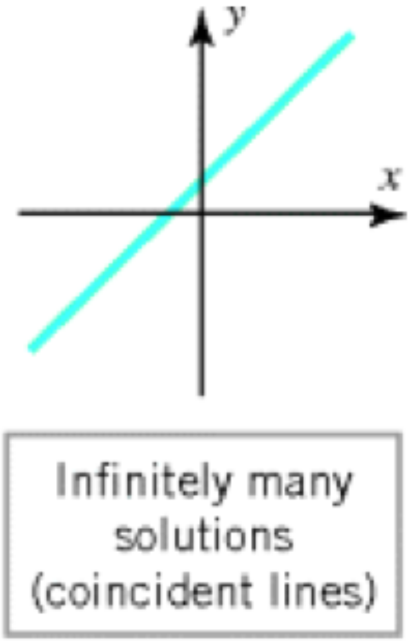
  - l1 and l2 may intersect at only one point: one solution



One solution

Linear Systems

- l1 and l2 may coincide: infinite many points of intersection, infinitely many solutions



Augmented Matrices & Elementary Row Operations

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Augmented Matrices

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

- This is called the augmented matrix for the system.
- For example the augmented matrix for the system of equations:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

- When constructing the unknowns must be written in the same order in each equation and the constants must be on the right.

Augmented Matrices

- The basic method for solving a liner system is:
  - Multiply an equation through by a nonzero constant
  - Interchange two equations
  - Add a constant times one equation to another

- Elementary row operation on a matrix is:
  - Multiply a row through by a nonzero constant
  - Interchange two rows
  - Add a constant times one row to another

Example #3

- Using Elementary Row Operations:

$$r_1: x + y + 2z = 9$$
$$r_2: 2x + 4y - 3z = 1$$
$$r_3: 3x + 6y - 5z = 0$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

- $r_2 = -2r_1 + r_2$

$$r_1: x + y + 2z = 9$$
$$r_2: 2y - 7z = -17$$
$$r_3: 3x + 6y - 5z = 0$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

- $r_3 = -3r_1 + r_3$

$$r_1: x + y + 2z = 9$$
$$r_2: 2y - 7z = -17$$
$$r_3: 3y - 11z = -27$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Example #3 (cont.)

- $r_2 = \frac{1}{2} r_2$

$$r_1: x + y + 2z = 9$$
$$r_2: y - \tfrac{7}{2} z = -\tfrac{17}{2}$$
$$r_3: 3y - 11z = -27$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

- $r_3 = -3r_2 + r_3$

$$r_1: x + y + 2z = 9$$
$$r_2: y - \tfrac{7}{2} z = -\tfrac{17}{2}$$
$$r_3: -\tfrac{1}{2} z = -\tfrac{3}{2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

- $r_3 = -2r_3$

$$r_1: x + y + 2z = 9$$
$$r_2: y - \tfrac{7}{2} z = -\tfrac{17}{2}$$
$$r_3: z = 3$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example #3 (cont.)

- $r_1 = r_1 - r_2$

$$r_1: x + \frac{11}{2}z = \frac{35}{2}$$
$$r_2: y - \frac{7}{2}z = -\frac{17}{2}$$
$$r_3: z = 3$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- $r1 = -11/2\ r_3 + r_1$
- $r_2 = 7/2\ r_3 + r_2$

$$r_1: x = 1$$
$$r_2: y - \frac{7}{2}z = -\frac{17}{2}$$
$$r_3: z = 3$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Solution:

$$r_1: x = 1$$
$$r_2: y = 2$$
$$r_3: z = 3$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Gaussian Elimination

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Echelon Forms

- For a reduced row echelon form a matrix must have the following properties:
  - If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leading 1.
  - If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
  - In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
  - Each column that contains a leading 1 has zeros everywhere else in that column.

- A matrix that has the first three properties is said to be in row echelon form.

Echelon Forms

- For example:
  - Reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  - Row-echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination Methods

- Elimination procedure can be used to reduce any matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step 1: Locate the leftmost nonzero column.
- Step 2: Interchange $r_2 \leftrightarrow r_1$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step 3: $r_1 = \frac{1}{2} r_1$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step 4: $r_3 = r_3 - 2r_1$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Elimination Methods (cont.)

- Step 5 : continue do all steps above until the entire matrix is in row-echelon form.

  - $r_2 = -\frac{1}{2} r_2$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

  - $r_3 = r_3 - 5r_2$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

  - $r_3 = 2r_3$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Elimination Methods (cont.)

- Step 6 : add suitable multiplies of each row to the rows above to introduce zeros above the leading 1's.

  - $r_2 = 7/2\, r_3 + r_2$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

  - $r_1 = -6r_3 + r_1$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

  - $r_1 = 5r_2 + r_1$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Elimination Methods (cont.)

- 1-5 steps produce a row-echelon form (Gaussian Elimination). Step 6 is producing a reduced row-echelon (Gauss-Jordan Elimination).
- Remark: Every matrix has a unique reduced row-echelon form, no matter how the row operations are varied. Row-echelon form of matrix is not unique: different sequences of row operations can produce different row- echelon forms.

Back-Substitution

- Bring the augmented matrix into row-echelon form only and then solve the corresponding system of equations by back-substitution.
- Example: [Solved by back substitution]

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

$$\text{Step 1.}\quad \begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Back-Substitution (cont.)

Step 2.

Substituting $x_6 = \frac{1}{3}$

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

Substituting $x_3 = -2x_4$

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

- Step 3. Assign arbitrary values to the free variables [parameters], if any

$$x_1 = -3r - 4s - 2t$$
$$x_2 = r$$
$$x_3 = -2s$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = \frac{1}{3}$$

Homogeneous Linear Systems

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Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if the constant terms are all zero; i.e., the system has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- Every homogeneous system of linear equations is consistent, since all such systems have $x_1=0, x_2=0, \ldots, x_n=0$ as a solution [trivial solution]. Other solutions are called nontrivial solutions.
- A homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:
  - The system has only the trivial solution.
  - The system has infinitely many solutions in addition to the trivial solution.

Example # 4

- Gauss-Jordan Elimination

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example # 4 (cont.)

- The corresponding system of equations is:

$$\begin{aligned} x_1 + x_2 + x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned}$$

- Solving for these variables yields:

$$\begin{aligned} x_1 &= -x_2 - x_5 \\ x_3 &= -x_5 \\ x_4 &= 0 \end{aligned}$$

- The general solution is:

$$x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$$

- The trivial solution is obtained when s=t=0.

Homogeneous Linear Systems

- Theorem:
  - A homogeneous system of linear equations with more unknowns than equation has infinitely many solutions.

Matrices & Matrix Operations

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Matrix Notation & Terminology

- A matrix is a rectangular array of numbers with rows and columns.
- The numbers in the array are called the entries in the matrix.
- Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \end{bmatrix}$
- The size of a matrix is described in terms of the number of rows and columns its contains.
- A matrix with only one column is called a column matrix or a column vector.
- A matrix with only one row is called a row matrix or a row vector.
- $a_{ij} = (A)_{ij}$ = the entry in row i and column j of a matrix A.

Matrix Notation & Terminology

- 1 x n row matrix a = [a_1 a_2 ... a_n]

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- m x 1 column matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- A matrix A with n rows and n columns is called a square matrix of order n. Main diagonal of A = {a_{11}, a_{22}, ..., a_{nn}}

Operations on Matrices

- Definition: **Equality of Matrices**
  - Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.
  - If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then A=B if and only if $(A)_{ij}=(B)_{ij}$, or equivalently $a_{ij}=b_{ij}$ for all i and j.
- Definition: **Addition & Subtraction**
  - If A and B are matrices of the same size, then the sum A+B is the matrix obtained by adding the entries of B to the corresponding entries of A.
  - The difference A-B is the matrix obtained by subtracting the entries of B from the corresponding entries of A.
  - Matrices of different sizes cannot be added or subtracted.
  - In matrix notation: If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then $(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$ and $(A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$

Operations on Matrices

- Definition: **Scalar Multiples**
  - If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a scalar multiple of A.
  - If $A = [a_{ij}]$, then $(cA)_{ij} = c(A)_{ij} = ca_{ij}$.
- Definition: **Multiplication of Matrices**
  - If A is an m x r matrix and B is an r x n matrix, then the product AB is the m x n matrix whose entries are determined as follows: To find the entry in row I and column j of AB, single out row I from the matrix A and column j from the matrix B.
  - Multiply the corresponding entries from the row and column together and then add up the resulting products.

Matrix Multiplication by Columns & by Rows

- Partitioning has many uses, one of which if for finding particular rows or columns of a matrix product AB without computing the entire product.
- For example: $AB = A\begin{bmatrix} b_1 & b_2 & .... & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & .... & Ab_n \end{bmatrix}$

$$(AB \quad computed \quad column \quad by \quad column)$$

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}$$

$$(AB \quad computed \quad row \quad by \quad row)$$

Matrix Products as Linear Combinations

- If A_1, A_2,….A_r are matrices of the same size and if c_1 , c_2,…. C_r are scalars, then an expression of the form:

$$c_1 A_1 + c_2 A_2 + \ldots\ldots + c_r A_r$$

- Is called a linear combination of A_1, A_2, …. A_r with coefficients c_1, c_2,…..,c_r.
- Let A be an m x n matrix and x an n x 1 column vector say:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Thus:

$$A = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix Products as Linear Combinations

- This proves the following theorem:
  - If A is an m x n matrix and if x is an n x 1 column vector, then the product Ax can be expressed a a linear combination of the column vectors of A in which the coefficients are the entries of x.

Matrix Form of a Linear System

- Matrix multiplication has an important applications to systems of linear equations.
- Consider a system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad\;\; \vdots \qquad & \;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

- Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix equation:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix Form of a Linear System

- The m x 1 matrix on the left side of this equation can be written as a product to give:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- If we designate these matrices by A, x and b respectively then we can replace the original system of m equations in n unknowns has been replaced by the single matrix equation. Ax=b.
- The matrix A in this equation is called the coefficient matrix of the system.

Matrix Form of a Linear System

- The augmented matrix of the system is obtained by adjoining b to A as the last column: thus augmented matrix is:

$$[A|b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- The vertical bar in [A|b] is a convenient way to separate A from b visually. It has no mathematical significance.

Transpose of a Matrix

- If A is any m x n matrix, then the transpose of A denoted by A^T is defined to be the n x m matrix that results by interchanging the row and columns of A; that is the first column of A^T is the first row of A, the second column of A^T is the second row of A and so forth.
- Mathematically can be written as: $(A^T)_{ij} = (A)_{ji}$
- For example some transposes are:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, B^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Trace of a Matrix

- If A is a square matrix, then the trace of A, denoted by tr(A) is defined to be the sum of the entries on the main diagonal of A. the trace of A is undefined if A is not a square matrix.
- For example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33}$$

Thankyou

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