Linear Algebra

System of Linear Equations & Matrices

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Inverses; Rules of Matrix Arithmetic

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Properties of Matrix Operations

- The following theorem lists the basic algebraic properties of the matrix operations.
- Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.
  - A+B = B+A (**Commutative law for addition**)
  - A+(B+C) = (A+B)+C (**Associative law for addition**)
  - A(BC) = (AB)C (**Associative law for multiplication**)
  - A(B+C) = AB+AC (**Left distributive law**)
  - (B+C)A = BA+CA (**Right distributive law**)
  - A(B-C) = AB-AC
  - (B-C)A = BA-CA
  - a(B+C) = aB+aC
  - a(B-C) = aB-aC

Properties of Matrix Operations

- $(a+b)C = aC+bC$
- $(a-b)C = aC-bC$
- $a(bC) = (ab)C$
- $a(BC) = (aB)C$

- To prove any of the equalities in this theorem we must show that the matrix on the left side has the same size as that on the right and that the corresponding entries on the two sides are the same.

Example #1

- Associativity of Matrix Multiplication:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

- Then:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Example #1 (cont.)

- Thus: $$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$
- And: $$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$
- So (AB)C = A(BC).

Properties of Matrix Multiplication

- In real arithmetic it is always true that ab=ba which is known as commutative law for multiplication. However, in matrix arithmetic the equality of AB and BA can fail for three possible reasons:
  - AB may be defined and BA may not. E.g.: if A is 2 x 3 and B is 3 x 4.
  - AB and BA may both be defined, but they may have different sizes. E.g.: if A is 2 x 3 and B is 3 x 4.
  - AB and BA may both be defined and have the same size, but the two matrices may be different.

Zero Matrices

- A matrix, all of whose entries are zero, such as

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [0]$$

- A zero matrix will be denoted by 0 or 0 mxn for the mxn zero matrix. 0 for zero matrix with one column.
- Properties of zero matrices:
- A + 0 = 0 + A = A
- A – A = 0
- 0 – A = -A
- A0 = 0; 0A = 0

Identity Matrices

- Square matrices with 1's on the main diagonal and 0's off the main diagonal, such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Notation: I_n = n x n identity matrix.
- If A = m x n matrix, then:
  - $AI_n = A$ and $I_nA = A$

Identity Matrices (cont.)

- To explain the role of identity matrices in matrix arithmetic, let us consider the effect of multiplying a general 2 x 3 matrix A on each side by an identity matrix. Multiplying in the right by the 3 x 3 identity matrix yields:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

- And multiplying on the left by 2 x 2 identity matrix yields:

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Identity Matrices (cont.)

- The same result holds in general; that is if A is any m x n matrix then: $AI_n = A$ and $I_mA = A$.
- Thus the identity matrices play the same role in these matrix equations that the number 1 plays in the numerical equation $a .1 = 1 . a = a$.

Inverse of a Matrix

- In real arithmetic every nonzero number a has the reciprocal a^{-1} i.e., 1/a, with the property: $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
- The number a^{-1} is sometimes called the multiplicative inverse of a.
- Definition: If A is a square matrix, and if a matrix B of the same size can be found such that AB=BA=I, then A is said to be invertible or non-singular and B is called an inverse of A . If no such matrix B can be found, then A is said to be singular.

Example #2

- An Invertible Matrix:
- Let:

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

- Then:

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Thus A and B are invertible and each is an inverse of the other.

Powers of a Matrix

- If A is a square matrix, then we define the nonnegative integer powers of A to be

  $A^0=I$ $A^n = AA\ldots A$ (n>0)

  n factors

- Moreover, if A is invertible, then we define the negative integer powers to be $A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\ldots\ldots A^{-1}$

  n factors

- Theorem: Laws of Exponents
  - If A is a square matrix, and r and s are integers, then $A^rA^s = A^{r+s} = A^{rs}$
  - If A is an invertible matrix, then
    - A^{-1} is invertible and $(A^{-1})^{-1} = A$
    - A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for n = 0, 1, 2, ...
    - For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = 1/k\ A^{-1}$.

Powers of a Matrix (cont.)

- Example:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Matrix Polynomials

- If A is a square matrix, say n x n, and if:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots.. + a_m x^m$$

- Is any polynomial, then we define the n x n matrix p(A) to be:

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \ldots.. + a_m A^m$$

- Where I is the n x n identity matrix that is p(A) is obtained by substituting A for x and replacing the constant term a_0 by the matrix a_0I. Above expression is called a matrix polynomial in A.

Example #3

- Find p(A) for: $p(x) = x^2 - 2x - 3 \quad and \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$
- Solution:

$$p(A) = A^2 - 2A - 3I$$

$$= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Or more briefly, p(A) = 0.

Properties of the Transpose

- Theorem: If the sizes of the matrices are such that the stated operations can be performed, then
  - $((A)^T)^T = A$
  - $(A+B)^T = A^T + B^T$ and $(A-B)^T = A^T - B^T$
  - $(kA)^T = kA^T$, where k is any scalar
  - $(AB)^T = B^T A^T$
- The transpose of a product of any number of matrices is equal to the product of their transpose in the reverse order.

Invertibility of a Transpose

- Theorem: If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
- For example:

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}, A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}, (A^{-1})^T = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}, (A^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

Elementary Matrices & Method for Finding A^{-1}

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Elementary Matrices

- Definition:
  - An n x n matrix is called an elementary matrix if it can be obtained from the n x n identity matrix I_n by performing a single elementary row operation.
- Example:

$$1:\begin{bmatrix}1 & 0\\0 & -3\end{bmatrix},2:\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix},3:\begin{bmatrix}1 & 0 & 3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

  - (1): Multiply the second row of I_2 by -3.
  - (2): Interchange the second and fourth rows of I_4.
  - (3): Add 3 times the third row of I_3 to the first row.

Row Operations by Matrix Multiplication

- If the elementary matrix E results from performing a certain row operation on I_m and if A is an m x n matrix, then the product EA is the matrix that results when this same row operation is performed on A.
- Example:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 0 \end{bmatrix}$$

  - EA is precisely the same matrix that results when we add 3 times the first row of A to the third row.

Row Operations by Matrix Multiplication

- If an elementary row operation is applied to an identity matrix I to produce an elementary matrix E, then there is a second row operation that, when applied to E produces I back again.
- Inverse Operation:

Row Operation on I that Produces E	Row Operation on E that Reproduces I
Multiply row i by c ≠ 0	Multiply row i by 1/c
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add –c times row i to row j

Elementary Matrices

- Theorem: Every elementary matrix is invertible and the inverse is also an elementary matrix.
- Theorem: Equivalent Statements
  - If A is an n x n matrix, then the following statements are equivalent, that is all true or all false:
    - A is invertible.
    - Ax=0 has only the trivial solution.
    - The reduced row echelon form of A is I_n.
    - A is expressible as a product of elementary matrices.

Method for Inverting Matrices

- To find the inverse of an invertible matrix, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

Example #4

- Find the inverse of : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$
- Solution:
  - Objective is to reduce A to the identity matrix by row operation and simultaneously apply these operations to I to produce A^{-1}.
  - Adjoin the identity matrix to the right side of A, thereby producing a matrix of the form [A|I]
  - Apply row operations to this matrix until the left side is reduced to I, so the final matrix will have the form $[I|A^{-1}]$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Example #4 (cont.)

- Add -2 times the first row to the second row and -1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

- We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

- Multiply the third row by -1:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Example #4 (cont.)

- Add 3 times the third row to the second and -3 times the third row to the first:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

- Add -2 times the second row to the first:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

- Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Diagonal, Triangular & Symmetric Matrices

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Diagonal Matrices

- A square matrix in which all the entries off the main diagonal are zero is called a diagonal matrix.
- For example:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

- A diagonal n x n diagonal matrix D can be written as:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \qquad D^k = \begin{bmatrix} d_1^{\,k} & 0 & \cdots & 0 \\ 0 & d_2^{\,k} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^{\,k} \end{bmatrix}$$

Triangular Matrices

- Lower triangular = a square matrix in which all the entries above the main diagonal are zero.
- Upper triangular = a square matrix in which all the entries under the main diagonal are zero.
- Triangular = a matrix that is either upper triangular or lower triangular.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Triangular Matrices (cont.)

- Example: $$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
  - The matrix A is invertible, since its diagonal entries are nonzero, but the matrix B is not.

$$A^{-1} = \begin{bmatrix} 1 & -3/2 & 7/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}, AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

  - This inverse is upper triangular.
  - This product is upper triangular.

Symmetric Matrices

- A square matrix A is called symmetric if $A=A^T$.

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

- A matrix $A=[a_{ij}]$ is symmetric if and only if $a_{ij} = a_{ji}$ for all values of i and j.

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

- A square matrix A is called skew-symmetric if $A^T= -A$.

Symmetric Matrices (cont.)

- Theorem: If A and B are symmetric matrices with the same size, and if k is any scalar, then
  - A^T is symmetric
  - A+B and A-B are symmetric
  - KA is symmetric
- Theorem:
  - If A is an invertible matrix, then A^{-1} is symmetric.
  - If A is an invertible matrix, then AA^T and A^TA are also invertible.

Thankyou

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