Linear Algebra

Determinants

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Determinants by Cofactor Expansion

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Determinant

- det(A) is a number where as A is a matrix.
- Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if ad – bc ≠0 and the expression ad – bc is called the determinant of the matrix A.
- Determinant is denoted by writing:

$$\det(A) = ad - bc \quad or \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- The inverse of A can be expressed in terms of the determinant as:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Minors & Cofactors

- If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the sub matrix that remains after the ith row and jth column are deleted from A.
- The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij} .

Example #1

- Let:

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

- The minor entry of a_{11} is:

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

- The cofactor of a_{11} is:

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

- Similarly, the minor of entry a_{32} is:

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

Example #1 (cont.)

- The cofactor of a_{32} is:

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

- Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either +1 or -1 in accordance with the pattern in the "checkerboard" array:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

General Determinant

- If A is an n x n matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sums themselves are called cofactor expansion of A.
- That is:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$
$$\left[\ cofactor\ \ \text{exp}ansion\ \ along\ \ the\ \ jth\ \ column\ \right]$$

- And:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$
$$\left[\ cofactor\ \ \text{exp}ansion\ \ along\ \ the\ \ ith\ \ row\ \right]$$

Example #2

- Evaluate det(A) by cofactor expansion along the first column of A by cofactor expansion along the first row:

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

- Solution:

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

Determinant of an Upper Triangular Matrix

- If A is an n x n triangular matrix (upper triangular, lower triangular or diagonal) then det (A) is the product of the entries on the main diagonal of the matrix, i.e., det (A)=$a_{11}a_{22}.....a_{nn}$.
- For example: The determinant of a 4 x 4 upper triangular matrix is the product of its diagonal entries. Each part of the computation uses a cofactor expansion along the first row.

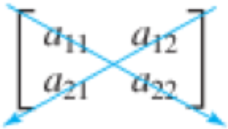
$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

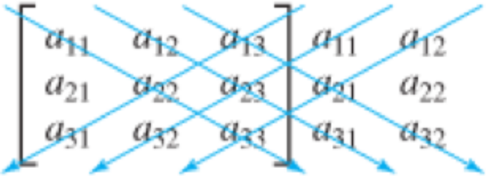
$$= a_{11}a_{22}\begin{vmatrix} a_{33} & 0 \\ a_{42} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}\left|a_{44}\right| = a_{11}a_{22}a_{33}a_{44}$$

Technique for Evaluating 2x2 & 3x3 Determinants

- Determinants of 2 x 2 and 3 x 3 matrices can be evaluated very efficiently using the pattern suggested below:





- The arrow technique only works for determinants of 2 x 2 and 3 x 3 matrices.

Example #3

- A technique for Evaluating 2 x 2 and 3 x 3 Determinants:

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

Evaluating Determinants by Row Reduction

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Basic Theorem

- Theorem 1: Let A be a square matrix. If A has a row of zeros or a column of zeros, then det (A)=0.
- Theorem 2: Let A be a square matrix. Then det (A) = det (A^T).
  - Because transposing a matrix changes its columns to rows and its rows to column, almost every theorem about the rows of a determinant has a companion version about columns and vice versa.

Elementary Row Operations

- Let A be an n x n matrix:
  - If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det (B) = k det (A).
  - If B is the matrix that results when two rows or two columns of A are interchanged, then det (B) = -det (A).
  - If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then det (B) = det (A).
- For example:

$$A = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$$

Elementary Row Operations (cont.)

Row Operation	Effect on Determinant
Interchange two rows	Change the sign
Multiply a row by a constant	Multiply by that constant
Add a multiple of a row to another row	No change

$$\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} = 10 - 6 = 4$$

$$\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 6 - 10 = -4$$

$$3 \times \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 3 & 6 \end{vmatrix} = 30 - 18 = 12 = 3(4)$$

$$\begin{vmatrix} 0 & -4 \\ 1 & 2 \end{vmatrix} = 0 - (-4) = 4$$

Elementary Matrices

- Let E be an n x n elementary matrix:
  - If E results from multiplying a row of I_n by a nonzero number k, then det (E) = k.
  - If E results from interchanging two rows of I_n , then det (E) = -1.
  - If E results from adding a multiple of on row of I_n to another, then det (E) = 1.

Example #4

- Determinants of Elementary Matrices:

Observe that the determinant of an elementary matrix cannot be zero.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

The second row of I_4 was multiplied by 3. **The first and last rows of I_4 were interchanged.** **7 times the last row of I_4 was added to the first row.**

Matrices with Proportional Rows or Columns

- If a square matrix A has two proportional rows, then a rows of zeros can be introduced by adding a suitable multiple of one of the rows to the other. Similarly for columns.
- But adding a multiple of one row or column to another does not change the determinant.
- Theorem: If A is a square matrix with two proportional rows or two proportional columns, then det (A) = 0.

Evaluating Determinants by Row Reduction

- This is a method for evaluating determinants that involves substantially less computation than cofactor expansion.
- We reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix, and then relate that determinant to that of the original matrix.

Example #5

- Using Row Reduction to Evaluate a Determinant:

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

- Solution:

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The first and second rows of A were interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

-2 times the first row was added to the third row.

Example #5 (cont.)

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

-10 times the second row was added to the third row.

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

Example #6

- Row Operations and Cofactor Expansion:

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

- Solution:
  - By adding suitable multiples of the second row to the remaining rows, we obtain:

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

Example #6 (cont.)

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

Cofactor expansion along the first column.

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

We added the first row to the third row.

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

Cofactor expansion along the first column.

$$= -18$$

Properties of Determinants; Cramer's Rule

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Basic Properties of Determinants

- det(A+B) ≠ det (A) + det (B).
- Theorem 1: Let A, B and C be n x n matrices that differ only in a single row, say the rth and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then, det (C) = det (A) + det (B).

Determinant Test for Invertibility

- Theorem 2: A square matrix A is invertible if and only if det (A) ≠ 0.
  - For example: Compute Det (A) where:

$$A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

  - Add 2 times row 1 to row 3 to obtain:

$$\det(A) = \det \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} = 0$$

  - Thus A is not invertible.
  - If A is invertible, then: $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$

Determinant of a Matrix Product

- If A and B are square matrices of the same size then:
  det (AB) = det (A) det (B)
- Also that if B is an n x n matrix and E is an n x n elementary matrix, then: det (EB) = det (E) det (B).

Example #7

- Verify above Theorem for:

$$A=\begin{bmatrix}6 & 1\\ 3 & 2\end{bmatrix} \quad and \quad B=\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}$$

- Solution:

$$AB=\begin{bmatrix}6 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}25 & 20\\ 14 & 13\end{bmatrix}$$

$$and$$

$$\det(AB)=25.13-20.14=325-280=45$$

$$\sin ce \quad \det(A)=9 \quad and \quad \det(B)=5$$

$$[\det(A)][\det(B)]=9.5=45=\det(AB)$$

Adjoint of a Matrix

- In a cofactor expansion we compute det (A) by multiplying the entries in a row or column by their cofactors and adding the resulting products.
- It turns out that if one multiplies the entries in any row by the corresponding cofactors from a different row, the sum of these products is always zero.
- Although the general proof is omitted, which is:

$$\det\left(A^{-1}\right) = \frac{1}{a_{11}}\frac{1}{a_{22}}\cdots\frac{1}{a_{nn}}$$

- Moreover, by using the adjoint formula it is possible to show that $\frac{1}{a_{11}}, \frac{1}{a_{22}}, \cdots, \frac{1}{a_{nn}}$ are actually the successive diagonal entries of A^{-1} .

Adjoint of a Matrix (cont.)

- If A is any n x n matrix and C_{ij} is the cofactor of a_{ij} , then the matrix:

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

- Is called the matrix of cofactor from A.
- The transpose of this matrix is called the adjoint of A and is denoted by adj (A).

Example #8

- Adjoint of a 3 x 3 matrix. Let: $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$
- The cofactor of A are:

$$C_{11} = 12 \quad C_{12} = 6 \quad C_{13} = -16$$

$$C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16$$

$$C_{31} = 12 \quad C_{32} = -10 \quad C_{33} = 16$$

- So the matrix of cofactors is:

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

- And the adjoint of A is:

$$adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Inverse of a Matrix Using its Adjoint

- If A is an invertible matrix, the:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- For example: det (A) = 64:

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Thankyou

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