Linear Algebra

Determinants

28th July 16

Cramer's Rule

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Cramer's Rule

- If Ax = b is a system of n linear equations in n unknowns such that det (A) ≠ 0, then the system has a unique solution.
- This solution is:

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots\ldots, x_n = \frac{\det(A_n)}{\det(A)}$$

- Where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example #1

- Solve a linear system using Cramer's rule:

$$\begin{aligned} x_1 & + & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{aligned}$$

- Solution:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Example #1 (cont.)

- Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Equivalence Theorem

- Theorem: Equivalent Statements
  - If A is an n x n matrix, then the following statements are equivalent:
    - A is invertible.
    - Ax=0 has only the trivial solution.
    - The reduced row echelon from of A is I_n.
    - A can be expressed as a product of elementary matrices.
    - Ax = b is consistent for every n x 1 matrix b.
    - Ax = b has exactly one solution for every n x 1 matrix b.
    - det (A) ≠ 0.

Example #2

- Use Cramer's rule to solve the system:

$$\begin{aligned} 3x_1 &- 2x_2 &= 6 \\ -5x_1 &+ 4x_2 &= 8 \end{aligned}$$

- Solution:

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \quad A_1 b = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \quad A_2 b = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

- Since det A=2, the system has a unique solution. By Cramer's rule:

$$x_1 = \frac{\det(A_1 b)}{\det(A)} = \frac{24+16}{2} = 20$$

$$x_2 = \frac{\det(A_2 b)}{\det(A)} = \frac{24+30}{2} = 27$$

Exercise Problems

28th July 16

Problem #1

- Verify that det (kA) = k^n det (A):

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}; k = -2$$

- Solution:

$$\det(A) = [20 - 1 + 36] - [6 + 8 - 15] = 56$$

$$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix}$$

$$= [-160 + 8 - 288] - [-48 - 64 + 120]$$

$$= -448 = (-2)^3 (56)$$

Problem #2

- Find the values of k for which A is invertible:

$$A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$$

- Solution:

$$\det(A) = (k-3)(k-2) - 4$$

$$= k^2 - 5k + 6 - 4 = k^2 - 5k + 2$$

  - Use the quadratic formula to solve:

$$k^2 - 5k + 2 = 0$$

$$k = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)} = \frac{5 \pm \sqrt{17}}{2}$$

$$A \quad is \quad invertible \quad for \quad k \neq \frac{5 \pm \sqrt{17}}{2}$$

Problem #3

- Decide whether the given matrix is invertible, and if so, use the adjoint method to find its inverse:

$$A=\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$$

- Solution:
  - Use row operations and cofactor expansion to simplify the determinant:

$$\det(A)=\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} -1 & 0 & 0 \\ 0 & 7 & 8 \\ 0 & 1 & 1 \end{vmatrix}=(-1)\begin{vmatrix} 7 & 8 \\ 1 & 1 \end{vmatrix}$$

$$=-(7-8)=1$$

Problem #3 (cont.)

- Since det (A) ≠ 0, A is invertible. The cofactors of A are:

$$C_{11} = \begin{vmatrix} 5 & 2 & 2 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -4$$

$$C_{12} = (-1)\begin{vmatrix} 2 & 2 & 2 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -(-2) = 2$$

$$C_{13} = \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = -7$$

$$C_{14} = (-1)\begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = -(-6) = 6$$

Problem #3 (cont.)

$$C_{21} = (-1)\begin{vmatrix} 3 & 1 & 1 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -(-3) = 3$$

$$C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -1$$

$$C_{23} = (-1)\begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = 0$$

$$C_{24} = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

$$C_{32} = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -1$$

$$C_{34} = (-1)\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -(-1) = 1$$

Problem #3 (cont.)

$$C_{41} = (-1)\begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = -(1) = -1$$

$$C_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 8 & 9 \end{vmatrix} = 0$$

$$C_{43} = (-1)\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 9 \end{vmatrix} = -(-8) = 8$$

$$C_{44} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{vmatrix} = -7$$

Problem #3 (cont.)

- The matrix of cofactors is:

$$\begin{bmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{1} \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$$

Problem #4

- Solve by using Cramer's rule:

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ 2x_1 - x_2 &= -2 \\ 4x_1 - 3x_3 &= 0 \end{aligned}$$

- Solution:

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix}$$

$$= (0+4) - 3(-1+6) = -11$$

Problem #4 (cont.)

$$A_1 = \begin{bmatrix} 4 & -3 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\det(A_1) = (-3)\begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix} = (-3)(-4-6) = 30$$

$$A_2 = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{bmatrix}$$

$$\det(A_2) = \begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} = (0+8) - 3(-2-8) = 38$$

$$A_3 = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{bmatrix}$$

$$\det(A_3) = 4\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = 4(6+4) = 40$$

Problem #4 (cont.)

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{30}{-11} = -\frac{30}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{38}{-11} = -\frac{38}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{40}{-11} = -\frac{40}{11}$$

$$The \quad solution \quad is \quad x_1 = -\frac{30}{11}, \quad x_2 = -\frac{38}{11}, \quad x_3 = -\frac{40}{11}$$

Problem #5

- Solve by using Cramer's rule:

$$\begin{aligned} 3x_1 &- x_2 &+ x_3 &= 4 \\ -x_1 &+ 7x_2 &- 2x_3 &= 1 \\ 2x_1 &+ 6x_2 &- x_3 &= 5 \end{aligned}$$

- Solution:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 7 & -2 \\ 2 & 6 & -1 \end{bmatrix}$$

  - Note that the third row of A is the sum of the first and second rows. Thus, det (A)=0 and Cramer's rule does not apply.

Problem #6

- In each part, find the determinant given that A is a 3 x 3 matrix for which det (A)=7:
  - (a): det (3A)
  - (b): det ($2A^{-1}$)
  - (c): det $(2A)^{-1}$
- Solution:
  - (a): det (3A):-
    - Using det (kA) = k^n det (A)

$$\det(3A) = 3^3 \det(A) = 27(7) \Rightarrow 189$$

  - (b): det ($2A^{-1}$):-

$$\det(2A^{-1}) = 2^3 \det(A^{-1}) = 8\left(\frac{1}{7}\right) \Rightarrow \frac{8}{7}$$

Problem #6 (cont.)

- (c): det $(2A)^{-1}$

$$\det\left((2A)^{-1}\right) = \frac{1}{\det(2A)} = \frac{1}{2^3 \det(A)}$$

$$= \frac{1}{8(7)} \Rightarrow \frac{1}{56}$$

Thankyou

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