Linear Algebra

Euclidean Vector Spaces

2nd Aug 16

Vectors in 2-Space, 3-Space & n-Space

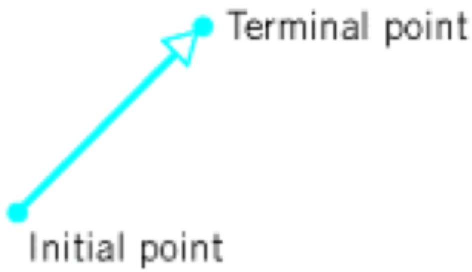
2nd Aug 16

Introduction

- There are two type of physical quantities:
  - Scalars
  - Vectors
- Scalars: are quantities that can be described by a numerical value alone.
- Vectors: are quantities that require both a number and a direction for their complete physical description.

Geometric Vectors

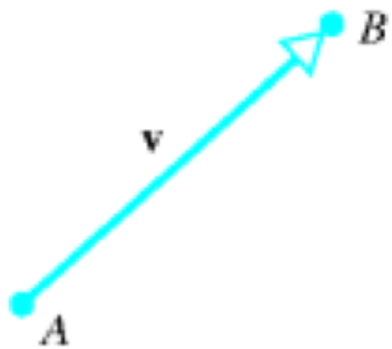
- Vectors are represented in two dimensions (also called 2-space) or in three dimension (also called 3-space) by arrows.
- The direction of the arrowhead specifies the direction of the vector and the length of the arrow specifies the magnitude.
- Mathematicians call these geometric vectors.
- The tail of the arrow is called the initial point of the vector and the tip the terminal point.



Geometric Vectors (cont.)

- Vectors will be represented in bold letters and scalars will be in italic type.
- A vector **v** has the initial point A and terminal point B as shown below:

$$\mathbf{v} = \overrightarrow{AB}$$



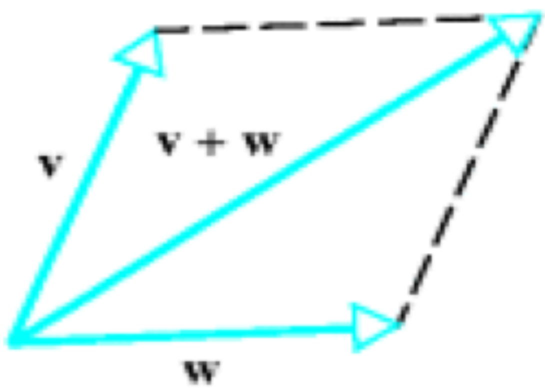
$$\mathbf{v} = \overrightarrow{AB}$$

Geometric Vectors (cont.)

- Vectors with same length and direction are said to be equivalent.
- Vector is determined solely by its length and direction, hence equivalent vectors are regarded to be the same vector even though they may be in different positions.
- Equivalent vectors are regarded to be the same vector even though they may be in different positions.
- The vector whose initial and terminal points coincide has length zero, so its known as zero vector and denote it by **0**.
- The zero vector has no natural direction, so we will agree that it can be assigned any direction that is convenient for the problem at hand.

Vector Addition

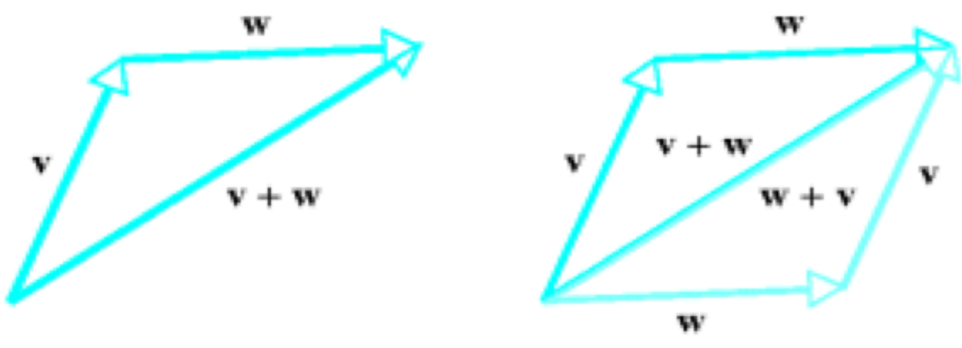
- **Parallelogram Rule for Vector Addition:**
  - If **v** and **w** are vectors in 2-space or 3-space that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram.
  - The sum **v+w** is the vector represented by the arrow from the common initial point of **v** and **w** to the opposite vertex of the parallelogram.



(a)

Vector Addition (cont.)

- **Triangle Rule for Vector Addition:**
  - If **v** and **w** are vectors in 2-space or 3-space that are positioned so the initial point of **w** is at the terminal point of **v**, then the sum **v+w** is represented by the arrow from the initial point of **v** to the terminal point of **w** (shown in figure b)
  - The sum **v+w** and **w+v** by the triangle rule is shown in figure c. This construction make it evident that: **v+w = w+v**



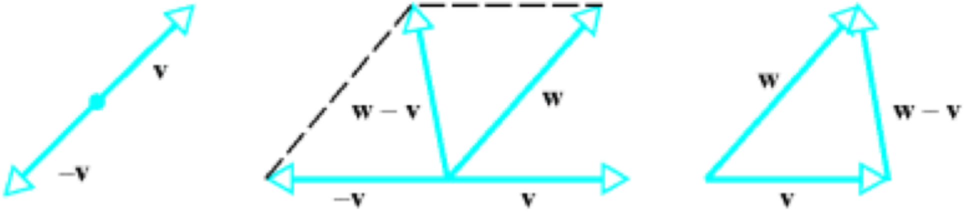
(b) (c)

Vector Addition (cont.)

- **Triangle Rule for Vector Addition: (cont.)**
  - The sum obtained by the triangle rule is the same as the sum obtained by the parallelogram rule.

Vector Subtraction

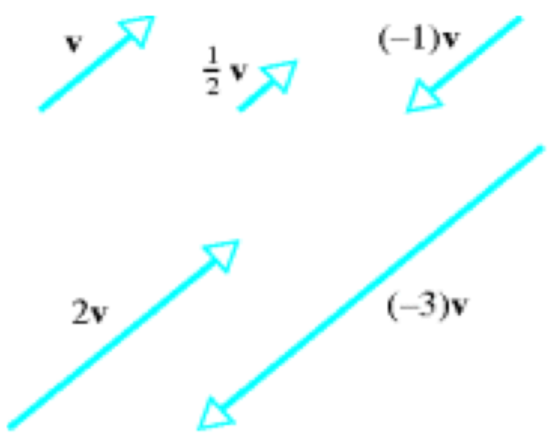
- The negative of a vector **v**, denoted by **–v** is the vector that has the same length as **v** but is oppositely directed. (shown in fig: a)
- The difference of **v** from **w**, denoted by **w-v** is taken to be the sum: **w-v = w + (-v)**.
- The difference of **v** and **w** can be obtained geometrically by the parallelogram method shown in figure b.
- Or more directly by positioning **w** and **v** so their initial points coincide and drawing the vector from the terminal point of **v** to the terminal point of **w** (shown in fig: c).



(a) (b) (c)

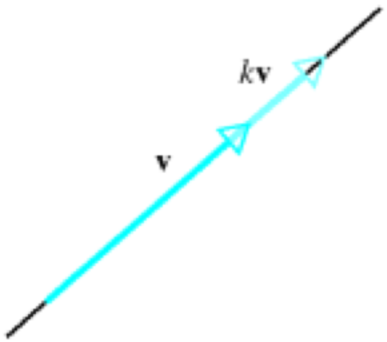
Scalar Multiplication

- If **v** is a nonzero vector in 2-space or 3-space and if k is a nonzero scalar, then we define the scalar product of **v** by **k** to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if k is positive and opposite to that of **v** if k is negative.
- If k=0 or **v**=**0**, then we define k**v** to be 0.
- The figure below shows the relationship between a vector **v** and some of its scalar multiple.



Parallel & Collinear Vectors

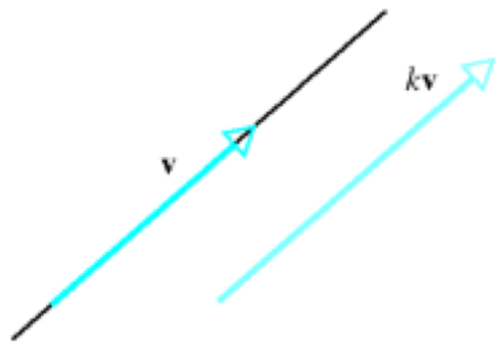
- Suppose that **v** and **w** are vectors in 2-space or 3-space with a common initial point.
- If one of the vectors is a scalar multiple of the other, then the vectors lie on a common line, so it is reasonable to say that they are collinear. (as shown below)



(a)

Parallel & Collinear Vectors (cont.)

- However, if we translate one of the vectors as indicated in fig b, then the vectors are parallel but no longer collinear.

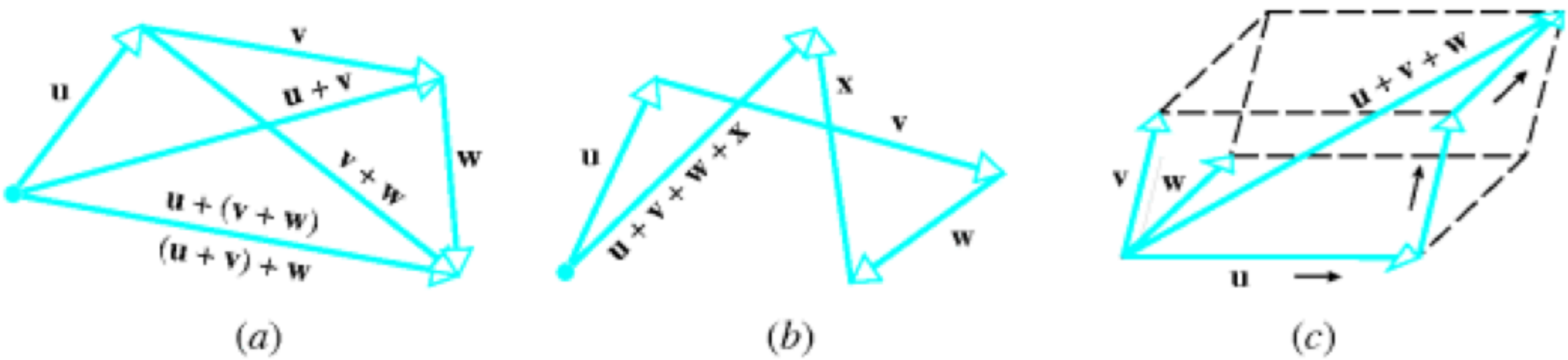


(b)

- Although the vector **0** has no clearly defined direction, we will regard it to be parallel to all vectors when convenient.

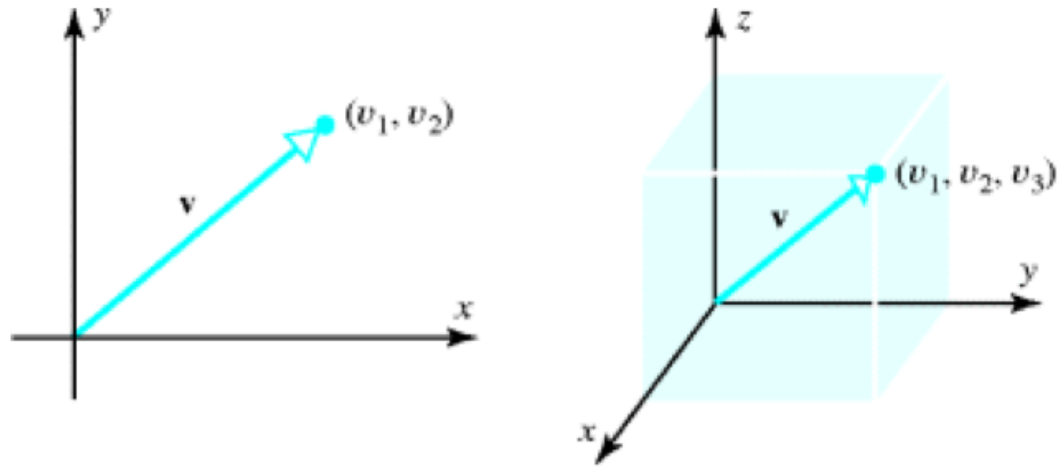
Sum of Three or More Vectors

- Vector addition satisfies the associative law for addition, meaning that when we add three vectors, say **u**, **v** , and **w** it does not matter which two we add first; i.e., **u + (v + w) = (u + v) + w** .
- A simple way to construct **u + v + w** is to place the vectors "tip to tail" in succession and then draw the vector from the initial point of **u** to the terminal point of **w**.



Vectors in Coordinate Systems

- If a vector **v** in 2-space or 3-space is positioned with its initial point at the origin of a rectangular coordinate system, then the vector is completely determined by the coordinates of its terminal point. As shown below:



- These coordinates are known as the components of **v** relative to the coordinate system.

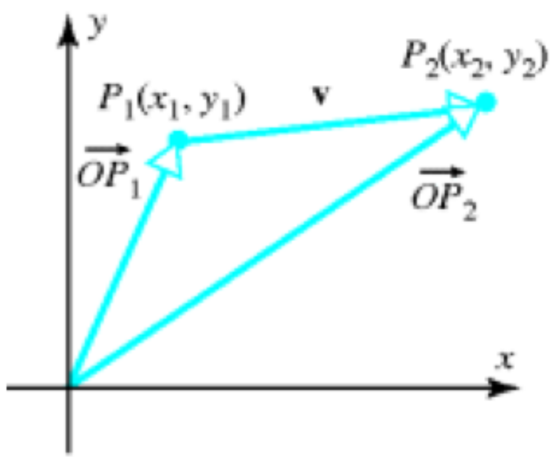
Vectors in Coordinate Systems (cont.)

- We will write **v** = (v_1 , v_2) to denote a vector **v** in 2-space with components (v_1 , v_2) and **v** = (v_1 , v_2 , v_3) to denote a vector **v** in 3-space with components (v_1 , v_2 , v_3) .
- The two vectors in 2-space or 3-space are equivalent if and only if they have the same terminal point when their initial points are at the origin.
- Algebraically, this means that two vectors are equivalent if and only if their corresponding components are equal.
- For example: **v** = (v_1 , v_2 , v_3) and **w** = (w_1 , w_2 , w_3) in 3-space are equivalent if and only if: $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$.

Vectors whose Initial Point is Not at the Origin

- If $\overrightarrow{P_1P_2}$ denotes the vector with initial point P_1 (x_1, y_1) and terminal point P_2 (x_2, y_2), then the components of this vector are given by the formula: $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$.
- The vector $\overrightarrow{P_1P_2}$ is the difference of vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$, so:

$$\overrightarrow{P_1P_2} = OP_1 - OP_2 = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$$



$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

Example #1

- Finding the components of a vector:
- The components of the vector **v** = $\overrightarrow{P_1P_2}$ with initial point P_1 (2 , -1 , 4) and the terminal point P_2 (7, 5, -8) are:

$$v = \left(7-2, 5-(-1)\right), \left(-8-4\right) = \left(5, 6, -12\right)$$

n-Space

- Definition: If n is an positive integer, then an ordered n-tuple is a sequence of n real numbers $(v_1 , v_2, ... , v_n)$. The set of all ordered n-tuples is called n-space and is denoted by R^n.

Operations on Vectors in R^n

- Definition: Vectors **v** = (v_1, v_2, …. , v_n) and **w** = (w_1 , w_2, …, w_n) in R^n are said to be equivalent if: $v_1 = w_1$, $v_2 = w_2$, $v_n = w_n$
- For example: Equality of vectors:

$$(a,b,c,d)=(1,-4,2,7)$$

$$If \quad and \quad only \quad if \quad a=1, \quad b=-4, \quad c=2 \quad and \quad d=7$$

Properties of Vector Operations

- Theorem: If **u** , **v** and **w** are vectors in R^n and if k and m are scalars, then:
  - (a): **u + v = v + u**
  - (b): **(u + v) + w = u + (v + w)**
  - (c): **u + 0 = 0 + u = u**
  - (d): **u + (-u) = 0**
  - (e): k **(u + v) =** k**u +** k**v**
  - (f): (k + m) **u** = k**u** + m**u**
  - (g): k(m**u**) = (km) **u**
  - (h): 1**u** = **u**
- Theorem: If **v** is a vector in R^n and k is a scalar, then:
  - (a): 0**v** = **0**
  - (b): k**0** = **0**
  - (c): (-1)**v** = **-v**

Linear Combinations

- Definition: If **w** is a vector in R^n, then **w** is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$,…., $\mathbf{v}_r$ in R^n if it can be expressed in the form: $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots. + k_r\mathbf{v}_r$.
- Where k_1, k_2 ,…, k_r are scalars. These scalars are called the coefficients of the linear combination.
- For example : If in case r=1, formula becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

Alternative Notations for Vectors

- A vector in R^n is just a list of its n components in a specific order any notation that displays those components in the correct order is a valid way of representing the vector.
- For example a vector in **v** = $[v_1 \ v_2 \ .... \ V_n]$ is called a row matrix form and in column matrix form it can be written as:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Norm, Dot Product, and Distance in R^n

2nd Aug 16

Norm of a Vector

- Definition: If **v** = (v_1, v_2, …., v_n) is a vector in R^n, then the norm of **v** (also called the length of **v** or the magnitude of **v**) is denoted by ||v|| and is defined by the formula:

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots.. + v_n^2}$$

Example #2

- Calculating Norms: vector **v** = (-3, 2, 1) in R^3 is:

$$\|v\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

- Norm of the vector **v** = (2, -1, 3, -5) in R^4 is:

$$\|v\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$

Theorem

- If **v** is a vector R^n, and if k is any scalar, then:
  - $||v|| \geq 0$
  - $||v|| = 0$ if and only if **v** =0
  - $||kv|| = |k|\ ||v||$

Unit Vectors

- A vector of norm 1 is called a unit vector.
- If **v** is any nonzero vector in R^n, then: $u = \frac{1}{\|v\|} v$ defines a unit vector that is in the same directions as **v**.
- The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called normalizing **v**.

Example #3

- Find the unit vector **u** that has the same direction as **v**= (2,2,-1).
- Solution:
  - The vector **v** has length:

$$\|v\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\textit{Thus},$$

$$u = \frac{1}{3}(2,2,-1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

The Standard Unit Vectors

- When a rectangular coordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinate axes are called the standard unit vectors.
- The standard unit vectors in R^n to be: $e_1 = (1,0,0,\ldots,0)$, $e_2=(0,1,0,\ldots,0)$,……, $e_n = (0,0,0,\ldots.,1)$ in which case every vector **v** = $(v_1, v_2, \ldots., v_n)$ in R^n can be expressed as:

$$v = (v_1, v_2, \ldots., v_n) = v_1 e_1 + v_2 e_2 + \ldots. + v_n e_n$$

Distance in R^n

- If **u** = (u_1, u_2,…,u_n) and **v** = (v_1, v_2,…,v_n) are points in R^n, then we denote the distance between **u** and **v** by d (**u**, **v**) and define it to be:

$$d(u,v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

$$d = \|\overrightarrow{P_1P_2}\|$$

Example #4

- Calculating Distance in R^n.
- If: **u** = (1,3,-2,7) and **v** = (0,7,2,2)
- Then the distance between **u** and **v** is:

$$d(u,v) = \|u - v\| = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2}$$

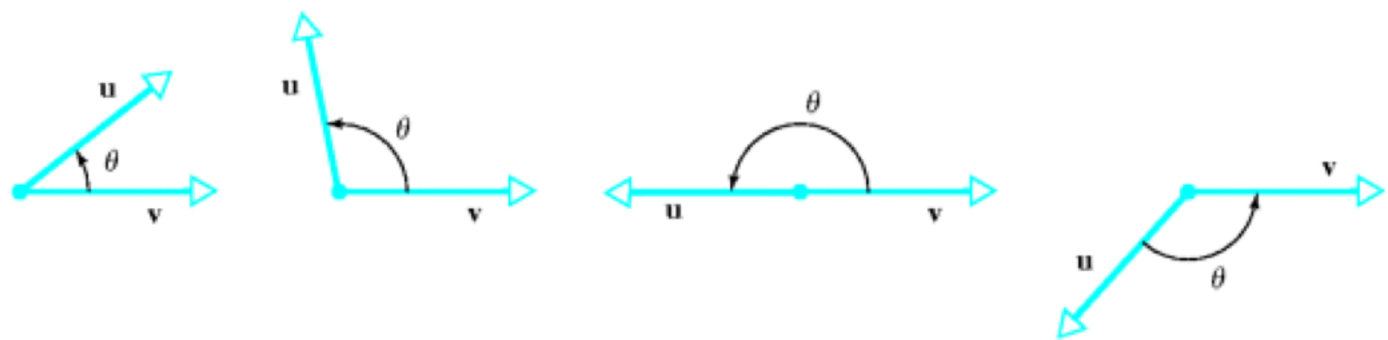
$$= \sqrt{58}$$

Dot Product

- If **u** and **v** are nonzero vectors in R^2 or R^3 and if θ is the angle between **u** and **v**, then the dot product also known as Euclidean inner product of **u** and **v** is denoted by **u.v** and is defined as:

$$u \cdot v = \|u\|\|v\|\cos\theta$$

- If **u = 0** or **v = 0**, then we define u . v to be 0.



The angle θ between **u** and **v** satisfies $0 \leq \theta \leq \pi$.

Dot Product (cont.)

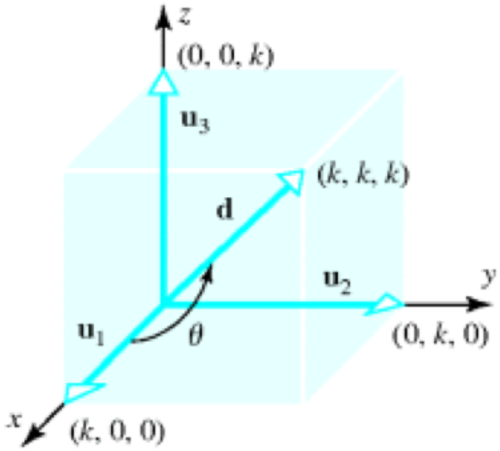
- Above formula of dot product can also be written as:

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

- Since $0 \leq \theta \leq \pi$, it follows from above formula and properties of the cosine function that:
  - θ is acute if u . v > 0
  - θ is obtuse if u . v < 0
  - $\theta = \pi/2$ if u . v =0

Example #5

- Find the angle between a diagonal of a cube and one of its edges.
- Solution:
  - Let k be the length of an edge and introduce a coordinate system as shown below.



  - If we let $\mathbf{u}_1$ = (k,0,0) , $\mathbf{u}_2$ = (0,k,0) and $\mathbf{u}_3$ = (0,0,k), then the vector: **d** = (k,k,k) = $\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3$ is a diagonal of the cube.
  - Then: $$\cos\theta = \frac{u_1 \cdot d}{\|u_1\|\|d\|} = \frac{k^2}{(k)\left(\sqrt{3k^2}\right)} = \frac{1}{\sqrt{3}}$$

Example #5 (cont.)

- With the help of a calculator we obtain:

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ$$

Component Form of the Dot Product

- If **u** = $(u_1, u_2, \ldots, u_n)$ and **v** = $(v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the dot product of **u** and **v** is denoted by **u . v** and is defined by:

  $$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Algebraic Properties of the Dot Product

- Theorem: If **u**, **v** and **w** are vectors in R^n and if k is a scalar, then:
  - (a): **u . v** = **v . u** [Symmetry Property]
  - (b): **u** . (**v + w**) = **u . v + u . w** [Distributive property]
  - (c)**:** k (**u . v**) = (k**u**) **. v** [Homogeneity property]
  - (d): **v . v** ≥ 0 and **v . v** = 0 if and only if **v** = 0 [Positivity property]

Dot Product as Matrix Multiplication

- If A is an n x n matrix and **u** and **v** are n x 1 matrices, then it follows that:

$$Au \cdot v = v^T(Au) = (v^T A)u = (A^T v)^T u = u \cdot A^T v$$

$$u \cdot Av = (Av)^T u = (v^T A^T)u = v^T(A^T u) = A^T u \cdot v$$

- The resulting formulas:

$$Au \cdot v = u \cdot A^T v$$

$$u \cdot Av = A^T u \cdot v$$

Dot Product as Matrix Multiplication

2^{nd} Aug 16

Table 1

Form	Dot Product	Example	
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

Dot Product as Matrix Multiplication

2nd Aug 16

$\mathbf{u}$ a column matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}$	$\mathbf{v}\mathbf{u}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$

Form	Dot Product	Example	
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}$	$\mathbf{u}\mathbf{v}^T=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

Thankyou

2nd Aug 16