Linear Algebra

Euclidean Vector Spaces

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Cauchy-Schwarz Inequality and Angles in R^n

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Cauchy-Schwarz Inequality

- If **u**= (u_1, u_2, …., u_n) and **v**= (v_1,v_2,….,v_n) are vectors in R^n, then

$$|u \cdot v| \leq \|u\| \|v\|$$

- Or in terms of components:

$$|u_1 v_1 + u_2 v_2 + \ldots. + u_n v_n| \leq \left(u_1^2 + u_2^2 + \ldots. + u_n^2\right)^{1/2} \left(v_1^2 + v_2^2 + \ldots. + v_n^2\right)^{1/2}$$

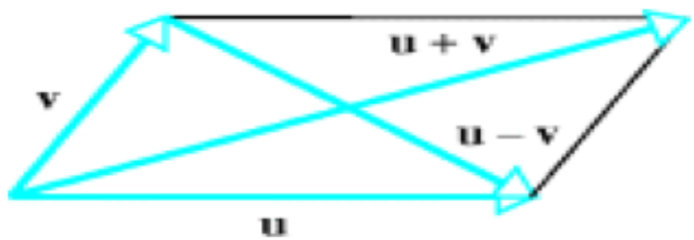
Geometry in R^n

- If **u** , **v** and **w** are vectors in R^n and if k is any scalar, then:
  - (a): $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ [Triangle inequality for vectors]
  - (b): $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ [Triangle inequality for distance]

Parallelogram Equation for Vectors

- If **u** and **v** are vectors in R^n, then:

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$



- Theorem: If **u** and **v** are vectors in R^n with the Euclidean inner product, then:

$$u \cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$

Orthogonality

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Orthogonal Vectors

- Two nonzero vectors **u** and **v** in R^n are said to be orthogonal (or perpendicular) if **u . v** =0.
- Also the zero vector in R^n is orthogonal to every vector in R^n. A nonempty set of vectors in R^n is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.
- An orthogonal set of unit vectors is called an orthonormal set.

Example #1

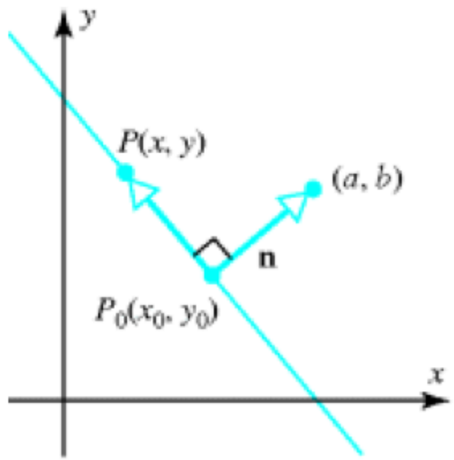
- (a): Show that **u** = (-2, 3, 1, 4) and **v** = (1, 2, 0, -1) are orthogonal vectors in R^4.
- (b): Show that the set S= (**i, j, k**) of standard unit vectors is an orthogonal set on R^3.
- Solution:
- (a): The vectors are orthogonal since,

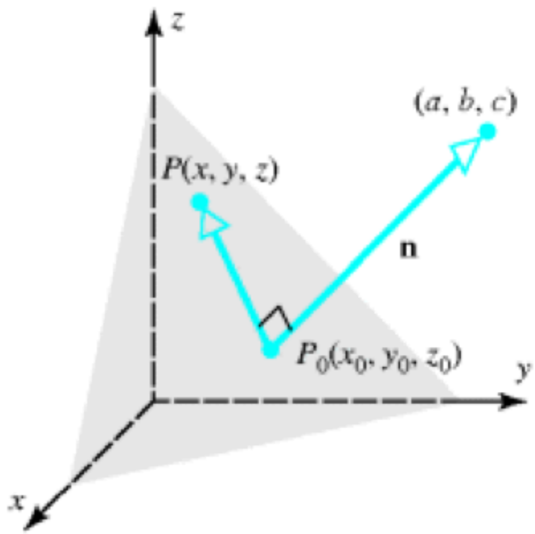
$$u \cdot v = (-2)(1) + (3)(3) + (1)(0) + (4)(-1) = 0$$

- (b): We must show that all pairs of distinct vectors are orthogonal, i.e., **i . j = i . k = j . k =** 0
- This is evident from the symmetry properties of the dot product.

Lines & Planes Determined by Points & Normals

- A line in R^2 is determined uniquely by its slope and one of its points, and a plane R^3 is determined uniquely by its "inclination" and one of its points.
- One way of specifying slope and inclination is to use a nonzero vector **n** called a normal, that is orthogonal to the line or plane .
- Figure below shows the line through the point $P_0(x_0 , y_0)$ that has normal **n** = (a, b) and the plane through the point P_0 (x_0 , y_0 , z_0) that has normal **n** = (a, b, c).





Lines & Planes Determined by Points & Normals (cont.)

- Both the line and the plane are represented by the vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.
- Where P is either an arbitrary point (x, y) on the line or an arbitrary point (x, y, z) in the plane.
- The vector $\overrightarrow{P_0P}$ can be expressed in terms of components as:

$$\overrightarrow{P_0P} = (x - x_0, y - y_0) \quad [Line]$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0) \quad [Plane]$$

$$a(x - x_0) + b(y - y_0) = 0 \quad [Line]$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad [Plane]$$

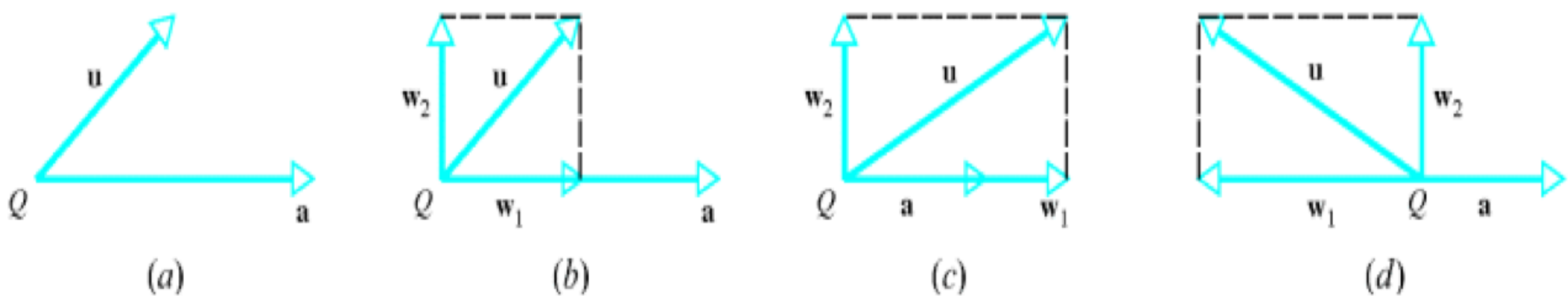
- These are called the point-normal equations of the line and plane.

Lines & Planes Determined by Points & Normals (cont.)

- Theorem:
  - If a and b are constants that are not both zero, then an equation of the form: $ax + by + c=0$ represents a line in R^2 with normal **n** = (a,b).
  - If a, b and c are constants that are not all zero, then an equation of the form: $ax + by +cz + d =0$ represents a plane in R^3 with normal = (a,b,c).

Orthogonal Projections

- If **u** and **a** are vectors in R^n , and if $a \neq 0$, then **u** can be expressed in exactly one way in the form $u = w_1 + w_2$, where w_1 is a scalar multiple of a and w_2 is orthogonal to a.
- The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the Projection Theorem have associated names: the vector $\mathbf{w}_1$ is called the orthogonal projection of u on a or sometimes the vector component of u along a, and the vector $\mathbf{w}_2$ is called the vector component of u orthogonal to a.



Orthogonal Projections

- The vector $\mathbf{w}_1$ is commonly denoted by the symbol $\text{proj}_{\mathbf{a}}\, \mathbf{u}$.

$$proj_a u = \frac{u \cdot a}{\|a\|^2} a \left(\text{ vector component of } u \text{ along } a \right)$$

$$u - proj_a u = u - \frac{u \cdot a}{\|a\|^2} a \left(\text{ vector component of } u \text{ orthognal to } a \right)$$

Example #2

- Find the orthogonal projections of the vectors e_1= (1,0) and e_2 = (0,1) on the line L that makes an angle θ with the positive x-axis in R^2.
- Solution:
  - **a=** (cos θ , sin θ) is a unit vector along the line L, so our first problem is to find the orthogonal projection of e_1 along **a** . Since:

$$\|a\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1 \quad and \quad e_1 \cdot a = (1,0)\cdot(\cos\theta, \sin\theta) = \cos\theta$$

  - From the formula of projection:

$$proj_a e_1 = \frac{e_1 \cdot a}{\|a\|^2} a = (\cos\theta)(\cos\theta, \sin\theta) = (\cos^2\theta, \sin\theta\cos\theta)$$

$$Similarly, \quad \sin ce \quad e_1 \cdot a = (0,1)\cdot(\cos\theta, \sin\theta) = \sin\theta$$

$$proj_a e_2 = \frac{e_2 \cdot a}{\|a\|^2} a = (\sin\theta)(\cos\theta, \sin\theta) = (\sin\theta, \cos\theta\sin^2\theta)$$

Theorem of Pythagoras

- If **u** and **v** are orthogonal vectors in R^n with the Euclidean inner product, then:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Example #3

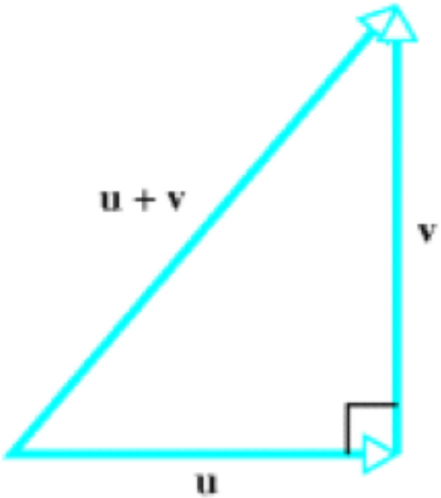
- Suppose we have vectors: **u**= (-2, 3, 1, 4) and **v** = (1,2,0,-1) are orthogonal. Verify the Theorem of Pythagoras for these vectors.
- Solution:

$$u + v = (-1, 5, 1, 3)$$

$$\|u + v\|^2 = 36$$

$$\|u\|^2 + \|v\|^2 = 30 + 6 = 36$$

$$Thus, \quad \|u + v\|^2 = \|u\|^2 + \|v\|^2$$



Thankyou

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