Linear Algebra

General Vector Spaces

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Row Space, Column Space & Null Space

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Definition

- If A is an m x n matrix, then the subspace of R^n spanned by the row vectors of A is called the row space of A.
- The subspace of R^m spanned by the column vectors of A is called the column space of A.
- The solution space of the homogeneous system of equations A**x**=0, which is a subspace of R^n, is called the null space of A.
- Theorem: A system of linear equations A**x** = **b** is consistent if and only if b is in the column space of A.

Example #1

- A vector **b** in the column space of A:
- Let A**x** = **b** be the linear system:

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

- Show that **b** is in the column space of A by expressing it as a linear combination of the column vectors of A.
- Solution:
- Solving the system by Gaussian elimination yields:

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

- It follows that:

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Theorem

- If $\mathbf{x}_0$ is any solution of a consistent linear system A**x** = **b**, and if S=($\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$) is a basis for the null space of A, then every solution of A**x** = **b** can be expressed in the form:

$$x = x_0 + c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

- Conversely, for all choices of scalars c_1, $c_2,\ldots,c_k$ the vector **x** in this formula is a solution of A**x** = **b**.
- The above equation gives a formula for the genral solution of A**x**= **b**.
- The vector $\mathbf{x}_0$ in that formula is called a particular solution of A**x** = **b** and the remaining part of the formulas is called the general solution of A**x** =0.

Bases for Row Spaces, Column Spaces & Null Spaces

- Theorem: Elementary row operations do not change the null space of a matrix.
- Elementary row operations do not change the row space of a matrix.
- Theorem: If a matrix R is in row echelon form, then the row vectors with the leading 1's form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

Example #2

- Basis for Row and Column Spaces:

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Is in row echelon form. The vectors:

$$r_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 \end{bmatrix}$$

$$r_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- Form a basis for the row space of R.

Example #2 (cont.)

- And the vectors:

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Form a basis for the column space of R.

Theorem

- If A and B are row equivalent matrices, then
  - (a): A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
  - (b): A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

Rank & Nullity

18th Aug 16

Row & Column Spaces have Equal Dimensions

- The row space and column space of a matrix A have the same dimension.

Rank & Nullity

- Definition: The common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by rank (A). The dimension of the null space of A is called the nullity of A and is denoted by nullity (A).

Example #3

- Find the rank and nullity of the matrix:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- Solution: The reduced row echelon form of A is:

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are two leading 1's, its row and column spaces are two dimensional and rank(A) = 2.
- Hence the corresponding system of equations will be:

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

Example #3 (cont.)

- Solving these equations for the leading variables yields:

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$
$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

- From which we obtain the general solution:

$$x_1 = 4r + 28s + 37t - 13u$$
$$x_2 = 2r + 12s + 16t - 5u$$
$$x_3 = r$$
$$x_4 = s$$
$$x_5 = t$$
$$x_6 = u$$

Example #3 (cont.)

- Or in column vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Because the four vectors on the right side of 3 form a basis for the solution space, nullity(A) = 4.

Example #4

- Maximum value of Rank:
- What is the maximum possible rank of m x n matrix A that is not square?
- Solution:
- Since the row vectors of A lie in R^n and the column vectors in R^m , the row space of A is at most n-dimensional and the column space is at most m-dimensional.
- Since the rank of A is the common dimension of its row and column space, it follows that the rank is at most the smaller of m and n.
- Denote this by writing: $rank(A) \le \min(m,n)$
- In which min (m,n) is the minimum of m and n.

Theorem

- Dimension theorem for matrices:
  - If A is a matrix with n columns, then

$$rank(A) + nullity(A) = n$$

Example #4

- The sum of Rank and Nullity:
- The matrix:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- Has 6 columns, so: $rank(A) + nullity(A) = 6$

$$rank(A) = 2, nullity(A) = 4$$

Theorem

- If A is an m x n matrix then,
  - (a): rank (A)= the number of leading variables in the general solution of Ax=0.
  - (b): nullity (A)= the number of parameters in the general solution of Ax=0.

Thankyou

18th Aug 16