Linear Algebra

Eigenvalues & Eigenvectors

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Eigenvalue & Eigenvector

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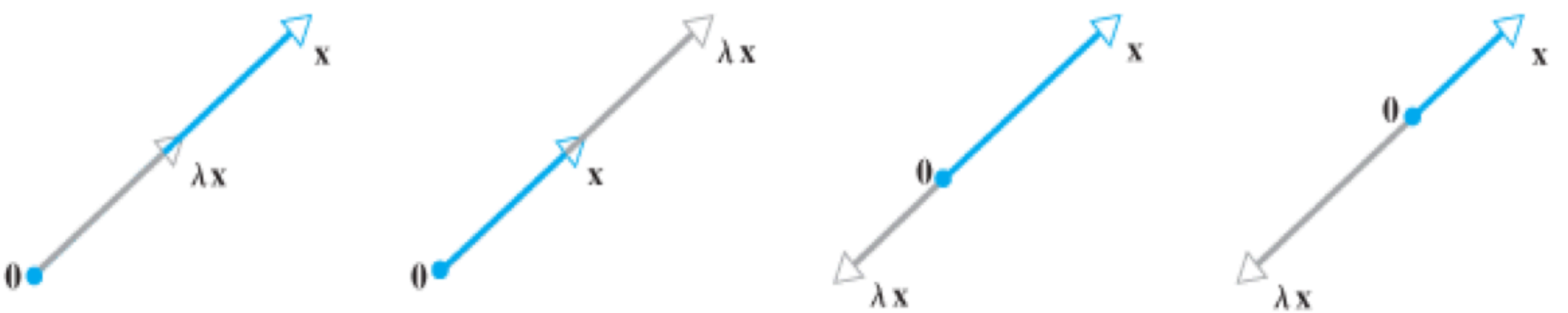
Definition

- If A is an n x n matrix, then a nonzero vector **x** in R^n is called an eigenvector of A (or of the matrix operator T_A) is A**x** is a scalar multiple of **x**; that is: A**x** = λ**x** for some scalar λ.
- The scalar λ is called an eigenvalue of A (or of T_A), and **x** is said to be an eigenvector corresponding to λ.
- The requirement that an eigenvector be nonzero is imposed to avoid the unimportant case A0 = λ0, which holds for every A and λ.



(a) $0 \le \lambda \le 1$ (b) $\lambda \le 1$ (c) $-1 \le \lambda \le 0$ (d) $\lambda \le -1$

Example #1

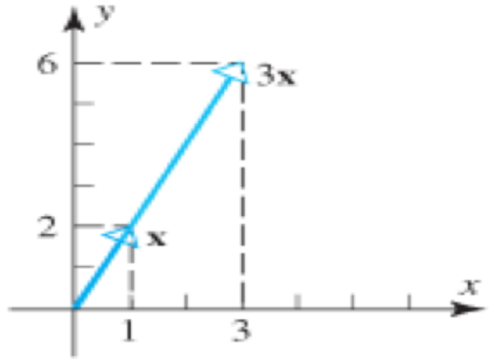
- Eigenvector of a 2 x 2 matrix:
- The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of:

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

- Corresponding to the eigenvalue λ=3, since

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3x$$

- Geometrically, multiplication by A has stretched the vector x by a factor of 3.



Computing Eigenvalues & Eigenvectors

- Theorem: If A is n x n matrix, then λ is an eigenvalue of A if and only if it satisfies the equation:

$$\det\left(\lambda I - A\right) = 0$$

- This is called the characteristic equation of A.

Example #2

- In Example 1 we observed that λ=3 is an eigenvalue of the matrix:

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

- But we did not explain how we found it. Use the characteristic equation to find all eigenvalues of this matrix.
- Solution:
- It follows from formula 1 that the eigenvalues of A are the solutions of the equation det (λI – A) = 0, which can be written as:

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

- From which we obtain: $(\lambda - 3)(\lambda + 1) = 0$

Example #2 (cont.)

- This shows that the eigenvalues of A are $\lambda=3$ and $\lambda=-1$. Thus, in addition to the eigenvalue $\lambda=3$ noted in example 1, we have discovered a second eigenvalue $\lambda=-1$.

Characteristic Polynomial

- When the determinant det (λI – A) that appears on the left side of 1 is expanded, the result is a polynomial p (λ) of degree n that is called the characteristic polynomial of A.
- In general, the characteristic polynomial of an n x n matrix has the form:

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n$$

- In which the coefficient of λ^n is 1. Since polynomial of degree n has at most n distinct roots, it follows that the equation:

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_n = 0$$

- Has at most n distinct solutions and consequently that an n x n matrix has at most n distinct eigenvalues.
- It is possible for a matrix to have complex eigenvalues, even if that matrix itself has real entries.

Example #3

- Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$
- Solution:
- The characteristic polynomial of A is:

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

- The eigenvalues of A must therefore satisfy the cubic equation:

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

- The only possible integer solutions of 4 are the divisors of -4, that is, ±1, ±2, ±4.

Example #3 (cont.)

- Substituting these values shows that λ= 4 is an integer solution.
- As a consequence, λ – 4 must be a factor of the left side of 4.
- Dividing λ – 4 into λ^3- 8 λ^2 + 17λ – 4 shows that 4 can be rewritten as: $$(\lambda-4)(\lambda^2-4\lambda+1)=0$$
- Thus, the remaining solutions of 4 satisfy the quadratic equation:

$$\lambda^2-4\lambda+1=0$$

- Which can be solved by the quadratic formula. Thus the eigenvalues of A are:

$$\lambda=4,\quad \lambda=2+\sqrt{3},\quad and\quad \lambda=2-\sqrt{3}$$

Theorem

- If A is an n x n triangular matrix (upper triangular, lower triangular or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.
- For example: By inspection, the eigenvalues of the lower triangular matrix:

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

$$are \quad \lambda = \frac{1}{2}, \quad \lambda = \frac{2}{3} \quad and \quad \lambda = -\frac{1}{4}$$

Theorem (cont.)

- If A is an n x n matrix, the following statements are equivalent:
  - λ is an eigenvalue of A.
  - The system of equations (λI – A) **x** = 0 has nontrivial solutions.
  - There is a nonzero vector **x** such that A**x** = λ**x** .
  - λ is a solution of the characteristic equation det (λI – A) =0.

Eigenvectors & Bases for Eigenspaces

23rd Aug 16

Finding Eigenvectors &Bases for Eigenspaces

- Since the eigenvectors corresponding to an eigenvalue λ of a matrix A are the nonzero vectors that satisfy the equation:

$$(\lambda I - A)x = 0$$

- These eigenvectors are the nonzero vectors in the null space of the matrix λI – A.
- This null space is known as the eigenspace of A corresponding to λ.
- The eigenspace of A corresponding to the eigenvalue λ is the solution space of the homogeneous system $(\lambda I - A)x = 0$.

Example #4

- Bases for Eigenspaces:
- Find the bases for the Eigenspaces of the matrix: $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$
- Solution:
- In example 1 we found the characteristic equation of A to be:

$$(\lambda - 3)(\lambda + 1) = 0$$

- From which we obtained the eigenvalues λ=3 and λ= -1. Thus, there are two Eigenspaces of A, one corresponding to each of these eigenvalues.
- By definition: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- Is an eigenvector of A corresponding to an eigenvalue λ if and only if **x** is a nontrivial solution of (λI – A) **x**= **0**, that is:

Example #4 (cont.)

$$\begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- If λ=3, then this equation becomes:

$$\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Whose general solution is: $x_1 = \frac{1}{2}t, \quad x_2 = t$
- Or in matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Thus: $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Example #4 (cont.)

- Is a basis for the eigenspace corresponding to λ=3,
- And $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to λ= -1 .

Powers of a Matrix

- Once the eigenvalues and eigenvectors of a matrix A are found, it is simple to find the eigenvalues and eigenvectors of any positive integer power of A; for example if λ is an eigenvalue of A and **x** is a corresponding eigenvector, then:

$$A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$$

- Which shows that λ^2 is an eigenvalue of A^2 and that **x** is a corresponding eigenvector.
- Theorem: If k is a positive integer, λ is an eigenvalue of a matrix A, and **x** is a corresponding eigenvector , then λ^k is an eigenvalue of A^k and **x** is a corresponding eigenvector.

Eigenvalues & Invertibility

- Theorem: A square matrix A is invertible if and only if $\lambda=0$ is not an eigenvalue of A.

Thankyou

23rd Aug 16