Linear Algebra

General Vector Spaces

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Real Vector Spaces

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Vector Space

- A vector space is a nonempty set V of objects, called vectors.
- Two operations are defined on vectors called addition and multiplication by scalars (real numbers), subject to ten axioms listed below: (The axioms must hold for all vectors **u**, **v** and **w** in V and for all scalars *c* and *d*.
  - The sum of **u** and **v**, denoted by **u+v**, is in V.
  - **u + v = v + u**
  - (**u + v**) + **w** = **u + (v + w**)
  - There is a **zero** vector **0** in V such that **u + 0 = u** .
  - For each **u** in V, there is a vector –**u** in V such that **u +** (-**u**) = **0**.
  - The scalar multiple of **u** by *c*, denoted by *c***u**, is in V.
  - *c* (**u** + **v**) = *c***u** + *c***v**.
  - (*c + d)* **u** = *c***u** + *d***u**
  - *c* (*d***u**) =(*cd*) **u**
  - 1**u** = **u**

Vector Space (cont.)

- For Example: The zero vector space
- Let V consist of a single object, which we denote by **0** and define:
- $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ for all scalars k.
- This is known as **zero vector space.**

Example #1

- A vector space of 2 x 2 Matrices:
- let V be the set of 2 x 2 matrices with real entries and take the vector space operations on V to be the usual operation of matrix addition and scalar multiplication.

$$u + v = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$ku = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

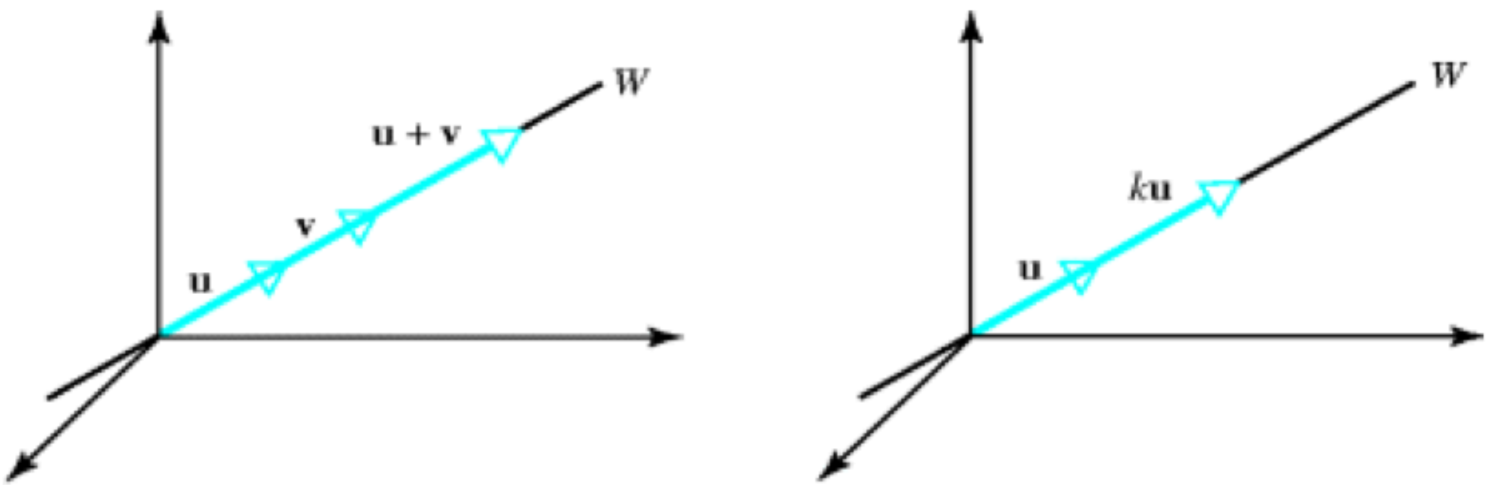
- The set V is closed under addition and scalar multiplication because the foregoing operations produce 2 x 2 matrices as the end result.
- Thus it remains to confirm that Axioms 2,3,4,5,7,8,9 and 10 hold.

Subspaces

- A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication.
- Theorem: If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions hold:
  - If **u** and **v** are vectors in W, then **u+v**, is in W.
  - If *k* is any scalar and **u** is any vector in W, then *k***u** is in W.

Lines Through the Origin

- Lines through the origin are subspaces of R^2 and R^3:
  - If W is a line through the origin of either R^2 or R^3, then adding two vectors on the line W or multiplying on the line W by a scalar produces another vector on the line W, so W is closed under addition and scalar multiplication.



(a) W is closed under addition.

(b) W is closed under scalar multiplication.

A Subspace Spanned by a Set

- Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let H= Span{$\mathbf{v}_1$, $\mathbf{v}_2$} . Show that H is a subspace of V:
- Solution:
- The zero vector is in H , since **0** = 0$\mathbf{v}_1$ + 0$\mathbf{v}_2$. To show that H is closed under vector addition, take two arbitrary vectors in H, say

$$u = s_1 v_1 + s_2 v_2 \quad and \quad w = t_1 v_1 + t_2 v_2$$

- By axioms 2,3, and 8 for the vector space v,

$$u + w = \left(s_1 v_1 + s_2 v_2\right) + \left(t_1 v_1 + t_2 v_2\right)$$

$$= \left(s_1 + t_1\right) v_1 + \left(s_2 + t_2\right) v_2$$

- So **u+w** is in H. Furthermore, if c is any scalar, then by Axioms 7 and 9:

$$cu = c\left(s_1 v_1 + s_2 v_2\right) = \left(cs_1\right) v_1 + \left(cs_2\right) v_2$$

A Subspace Spanned by a Set

- Example continued:
- Which shows that c**u** is in H and H is closed under scalar multiplication. Thus H is a subspace of V.
- Theorem: If $\mathbf{v}_1,\ldots\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots\ldots,\mathbf{v}_p\}$ is a subspace of V.

Example #2

- Let H be the set of all vectors of the form (a-3b, b-a, a, b), where a and b are arbitrary scalars. That is , let H={(a-3b, b-a, a, b) : a and b in R} . Show that H is a subspace of R^4.
- Solution:
- Write the vectors in H as column vectors. Then an arbitrary vector in H has the form:

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\uparrow_{v_1} \qquad \uparrow_{v_2}$$

- This calculation shows that H = Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors indicated above. Thus H is a subspace of R^4 by Theorem1.

Definitions

- Definition 1: If **w** is a vector space V, then **w** is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_r$ in V if **w** can be expressed in the form:

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

- Where k_1, k_2,…., k_r are scalars. These scalars are called the coefficients of the linear combination.
- Definition 2: the subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the span of S and we say that the vectors in S span that subspace. If S= {$\mathbf{w}_1$, $\mathbf{w}_2$, …, $\mathbf{w}_r$} , then we denote the span of S by:

$$span\{w_1, w_2, \ldots, w_r\} \quad or \quad span(S)$$

Example #3

- Consider the vectors **u** = (1,2,-1) and **v** = (6,4,2) in R^3. Show that **w** = (9,2,7) is a linear combination of **u** and **v** and that w' = (4,-1,8) is not a linear combination of **u** and **v**.
- Solution:
- In order for **w** to be linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that **w** = k_1**u** + k_2**v** ; that is:

$$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$$

$$or$$

$$(9,2,7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

- Equating corresponding components gives:

$$k_1 + 6k_2 = 9$$
$$2k_1 + 4k_2 = 2$$
$$-k_1 + 2k_2 = 7$$

Example #3 (cont.)

- Solving this system using Gaussian elimination yields k_1= -3 , k_2 = 2, so:

$$w = -3u + 2v$$

- Similarly, for **w'** to be a linear combination **u** and **v**, there must be scalars k_1 and k_2 such that **w'** = k_1**u** + k_2**v**; that is,

$$(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$$

$$or$$

$$(4,-1,8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

- Equating corresponding components gives:

$$k_1 + 6k_2 = 4$$

$$2k_1 + 4k_2 = -1$$

$$-k_1 + 2k_2 = 8$$

- This system of equations is inconsistent, so no such scalars k_1 and k_2 exist. Consequently, **w'** is not a linear combination of **u** and **v**.

Example #4

- Determine whether $\mathbf{v}_1 = (1,1,2)$, $\mathbf{v}_2 = (1,0,1)$ and $\mathbf{v}_3 = (2,1,3)$ span the vector space R^3 .
- Solution:
- We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination: $\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Expressing this equation in terms of components gives:

$$(b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$

$$or$$

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

$$or$$

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

Example #4 (cont.)

- Thus, our problem reduces to ascertaining whether this system is consistent for all values of b_1 , b_2, and b_3.
- One way of doing this is to use the fact that the system is consistent if and only if its coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

- Has a nonzero determinant.

Thankyou

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