Linear Algebra

General Vector Spaces

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Linear Independence

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Example

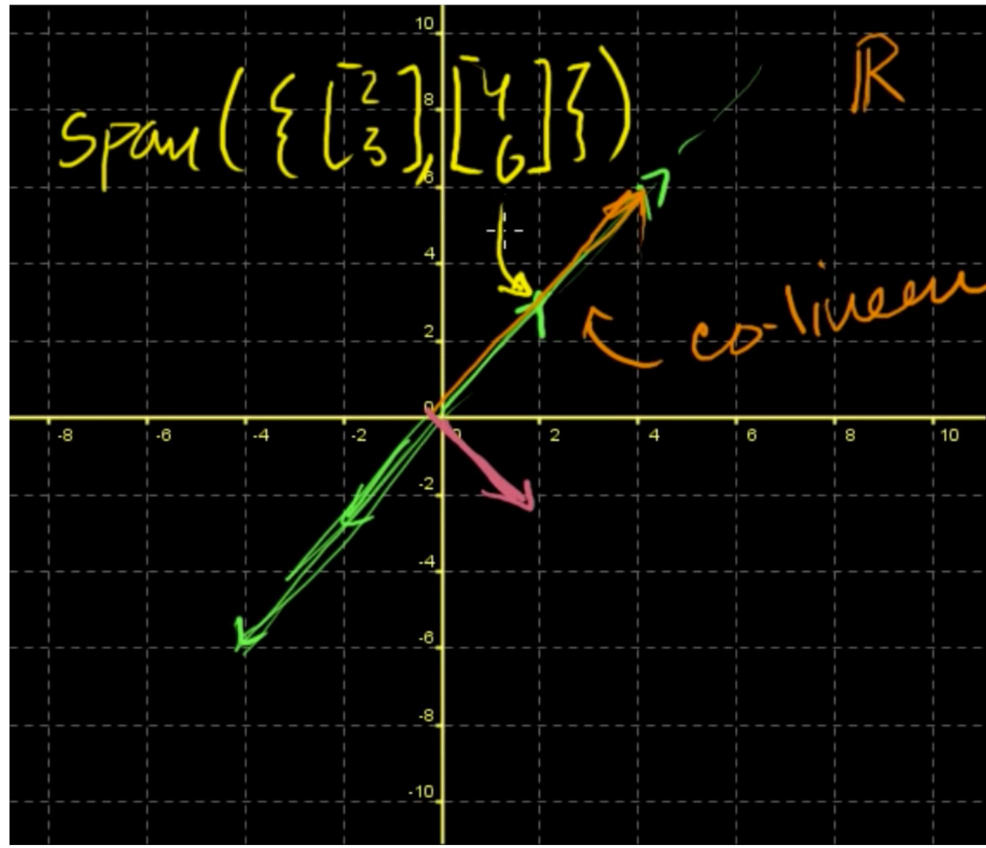
- Suppose we have two vectors:

$$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$(c_1 + c_2 2) \begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Linearly Independence & Dependence

- Definition: If $S=(v_1, v_2,.....,v_r)$ is a nonempty set of vectors in a vector space V, then the vector equation $k_1v_1 + k_2v_2 +....+ k_rv_r = 0$, has at least one solution namely: $k_1= 0$, $k_2 = 0$,..., $k_r= 0$.
- This is known as the trivial solution.
- If this is the only solution, then S is said to be a linearly independent set.
- If there are solution in addition to the trivial solution, then S is said to be a linearly dependent set.

Example #1

- Determine whether the vectors: $\mathbf{v}_1$= (1,-2,3), $\mathbf{v}_2$= (5,6,-1), $\mathbf{v}_3$= (3,2,1) are linearly independent or linearly dependent in R^3.
- Solution:
  - The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solution of the vector equation: $k_1v_1 + k_2v_2 + k_3v_3 = 0$ or equivalently of $k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$
  - Equating corresponding components on two sides yields the homogeneous linear system:

$$k_1 + 5k_2 + 3k_3 = 0$$
$$-2k_1 + 6k_2 + 2k_3 = 0$$
$$3k_1 - k_2 + k_3 = 0$$

  - Thus our problem reduces to determining whether this system has non trivial solutions. There are various ways to do this; one possibility is to simply solve the system which yields:

$$k_1 = -\frac{1}{2}t, k_2 = -\frac{1}{2}t, k_3 = t$$

Example #1 (cont.)

- Omit the details. This shows that the system has non trivial solutions and hence that the vectors are linearly dependent.
- Second method for obtaining the same result is to compute the determinant of the coefficient matrix.
- If the det (A) = 0, then there will be non trivial solutions and the vectors will be linearly dependent.

Spanning Vectors

- Definition: The vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$ in V span V if every vector $v \in V$ is a linear combination $k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n$ of $\mathbf{v}_1,\ldots, \mathbf{v}_n$.

Example #2

- Test the following vectors for linear independence?

$$u_1 = (1,0,0,1), u_2 = (1,0,1,0), u_3 = (0,1,1,0)$$

- We set: $x_1(1,0,0,1) + x_2(1,0,1,0) + x_3(0,1,1,0) = (0,0,0,0)$

- This vector equation really four equations:

$$x_1 + x_2 = 0$$

$$x_3 = 0$$

$$x_2 + x_3 = 0$$

$$x_1 = 0$$

- The only solutions are: $x_1 = x_2 = x_3 = 0$
- So the vectors (1,0,0,1), (1,0,1,0), (0,1,1,0) are Linearly Independent.

Alternative Interpretation of Linear Independence

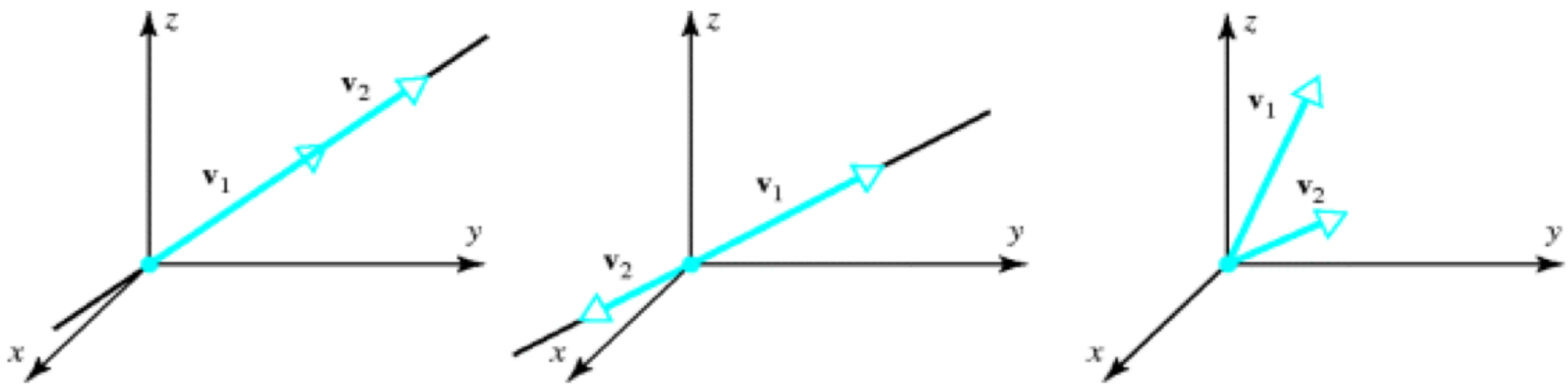
- Theorem: A set S with two or more vectors is:
  - (a): Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the vectors in S.
  - (b): Linearly independent if and only if no vectors in S is expressible as a linear combination of the other vectors in S.

Sets with One or Two vectors

- Theorem:
  - (a): a finite set that contains **0** is linearly dependent.
  - (b): A set with exactly one vector is linearly independent if and only if that vector is not 0.
  - (c): A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Geometric Interpretation of Linear Independence

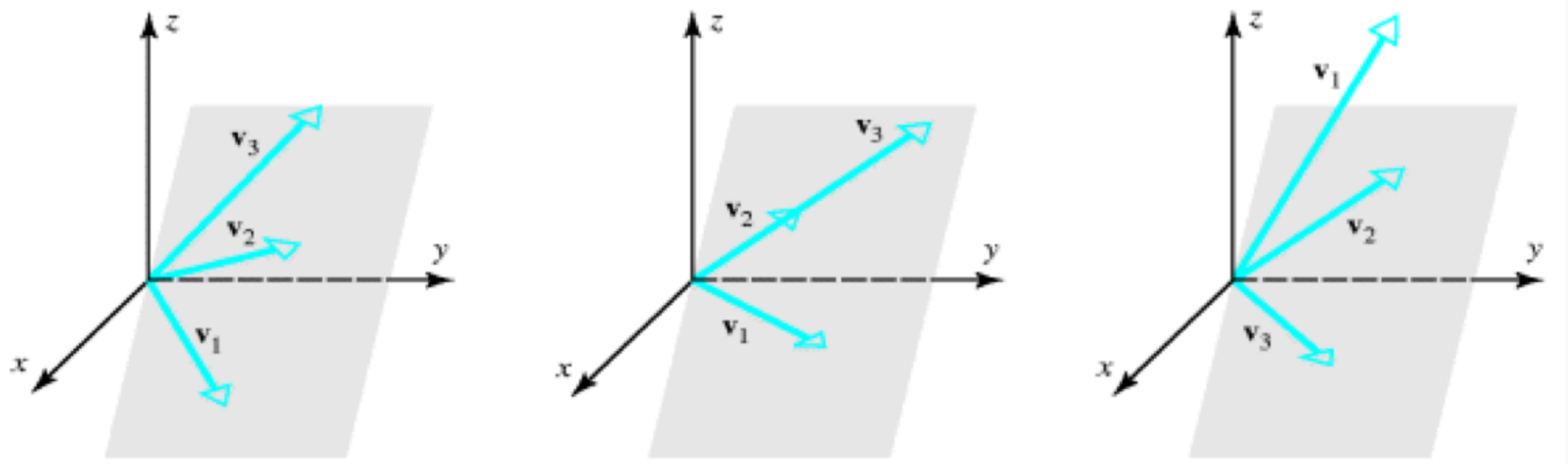
- Linear independence has the following useful geometric interpretations in R^2 and R^3:
  - Two vectors in R^2 or R^3 are linearly independent if and only if they do not lie on the same line when they have their initial points at the origin. Otherwise one would be a scalar multiple of the other.



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

Geometric Interpretation of Linear Independence (cont.)

- Three vectors in R^3 are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Basis for a Vector Space

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Basis

- Definition: If V is any vector space and S= ($\mathbf{v}_1$, $\mathbf{v}_2$,…, $\mathbf{v}_n$) is a finite set of vectors in V, then S is called a basis for V if the following two conditions hold:
  - (a): S is linearly independent.
  - (b) S spans V.

Example #3

- The standard basis for R^n.
- Suppose that the standard unit vectors:

$$e_1 = (1,0,0,...,0), e_2 = (0,1,0,....,0), e_n = (0,0,....,1)$$

- Span R^n and they are linearly independent. Thus, they form a basis for R^n that we call the standard basis for R^n.
- In particular, **i**= (1,0,0), **j**= (0,1,0), **k**= (0,0,1) is the standard basis for R^3.

Example #4

- The standard basis for M_{mn}:
- Show that the matrices:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Form a basis for the vector space M_{22} of 2 x 2 matrices:

- Solution:
  - We must show that the matrices are linearly independent and span M_{22} . To prove linear independence we must show that the equation:

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = 0$$

  - Has only the trivial solution, where **0** is the 2 x 2 zero matrix; and to prove that the matrices span M_{22} we must show that every 2 x 2 matrix:

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Example #4 (cont.)

- Can be expressed as:

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B$$

- The matrix forms of both the equations are:

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Which can be rewritten as:

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad and \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Example #4 (cont.)

- Since the first equation has only the trivial solution:

$$c_1 = c_2 = c_3 = c_4 = 0$$

- The matrices are linearly independent and since the second equation has the solution: $c_1 = a, c_2 = b, c_3 = c, c_4 = d$

- The matrices span M_{22} . This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}.
- More generally, the mn different matrices whose entries are zero except for a single entry of 1 form a basis for M_{mn} called the standard basis for M_{mn} .

Thankyou

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