Signal & Systems

Continuous & Discrete Signals

19TH October 16

Classification of Signals

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Continuous-Time Complex Exponential

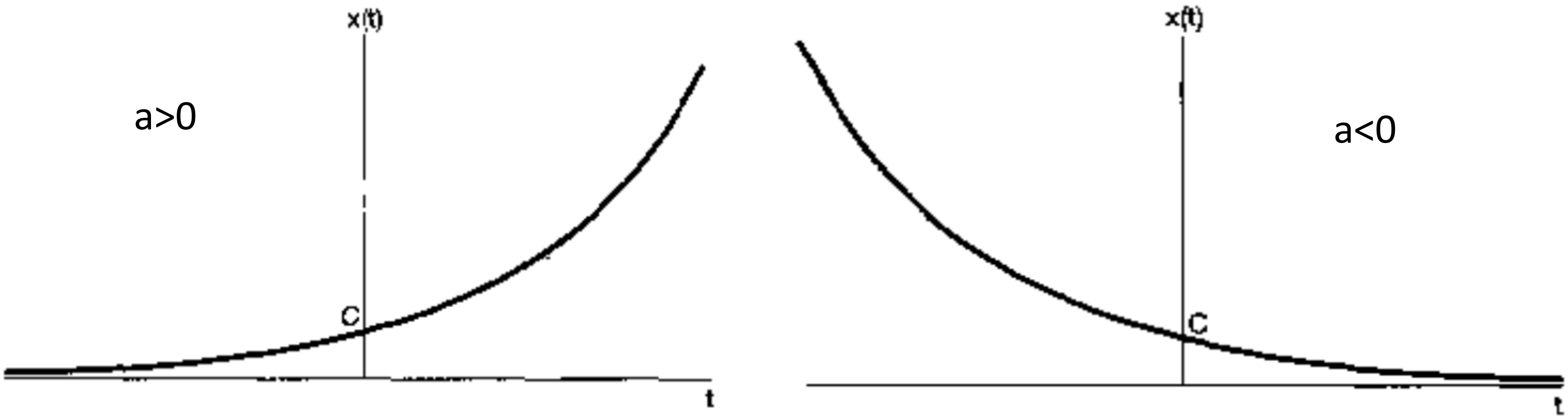
- The continuous-time complex exponential signal is of the form:

$$x(t) = Ce^{at}, \quad where \quad C, \quad a \in C$$

- Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

Real Exponential Signals

- If C and a are real there are basically two types of behaviour.
- If a is positive, then as t increase x(t) is a growing exponential, i.e., when a>0.
- If a is negative then x(t) is a decaying exponential, i.e., when a<0.
- When a=0 then x(t) is constant.



Periodic Complex Exponential

- Let's consider the case where a is purely imaginary, i.e., a = $j\omega_0$, ω_0 belongs to R.
- Since C is a complex number, we have: $C = Ae^{j\theta}$ where A, θ belongs to R.
- Consequently: $x(t) = Ce^{j\omega_0 t} = Ae^{j\theta}e^{j\omega_0 t}$

$$= Ae^{j(\omega_0 t+\theta)} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)$$

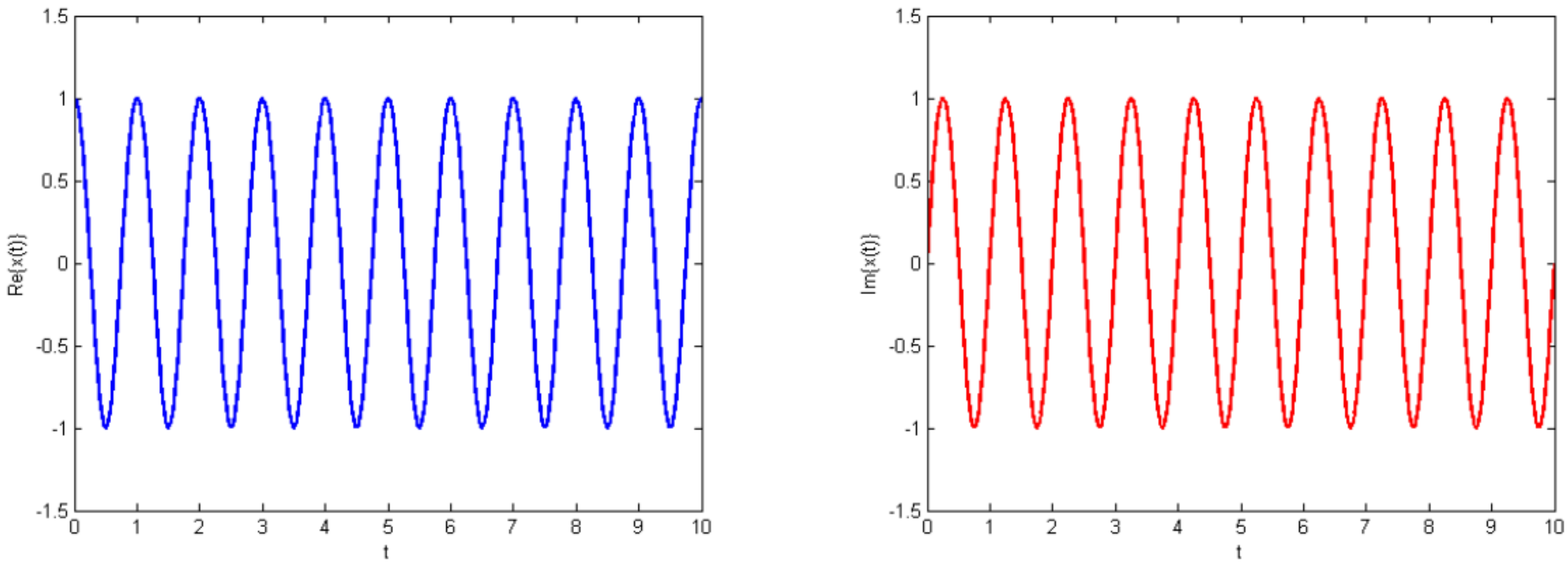
- The real and imaginary parts of x(t) are:

$$\mathrm{Re}\{x(t)\} = A\cos(\omega_0 t + \theta)$$

$$\mathrm{Im}\{x(t)\} = A\sin(\omega_0 t + \theta)$$

Periodic Complex Exponential (cont.)

❖ We can think of x(t) as a pair of sinusoidal signals of the same amplitude A, ω_0 and phase shift θ with one a cosine and the other a sine.



(a) $\mathcal{R}e\{Ce^{j\omega_0 t}\}$ (b) $\mathcal{I}m\{Ce^{j\omega_0 t}\}$

Periodic complex exponential function $x(t)= Ce^{j\omega 0t}$, $C=1$, $\omega_0=2\pi$

Periodic Complex Exponential (cont.)

- $x(t) = Ce^{j\omega_0 t}$ is periodic with:
  - Fundamental period: $T_0 = 2\pi/|\omega_0|$
  - Fundamental frequency: $|\omega_0|$
- the second claim is the immediate result from the first claim. To show the first claim, we need to show that $x(t+T_0) = x(t)$ and no smaller T_0 can satisfy the periodicity criteria.

$$x(t+T_0) = Ce^{j\omega_0\left(t+\frac{2\pi}{|\omega_0|}\right)} = Ce^{j\omega_0 t}e^{\pm j2\pi}$$

$$= Ce^{j\omega_0 t} = x(t)$$

- It is easy to show that T_0 is the smallest period.

General Complex Exponential

- The most general case of a complex exponential can be expressed and interpreted in terms of the two cases: the real exponential and the periodic complex exponential.
- Consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. I.e.,

$$C = |C|e^{j\theta}$$

- And:

$$a = r + j\omega_0$$

- Then:

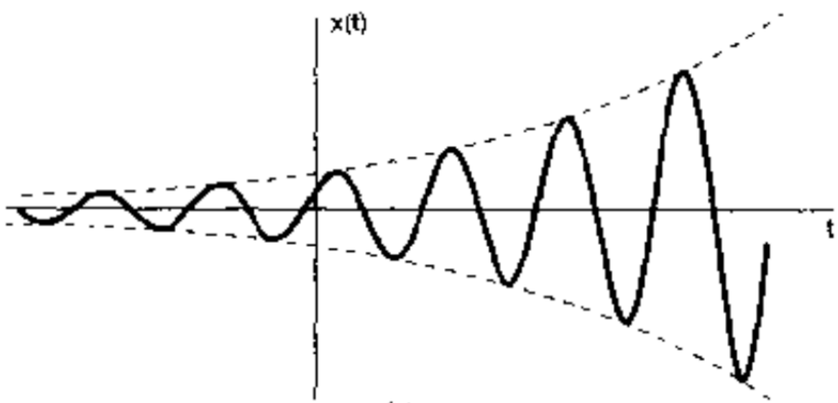
$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

- Using Euler's relation, we can expand this further as:

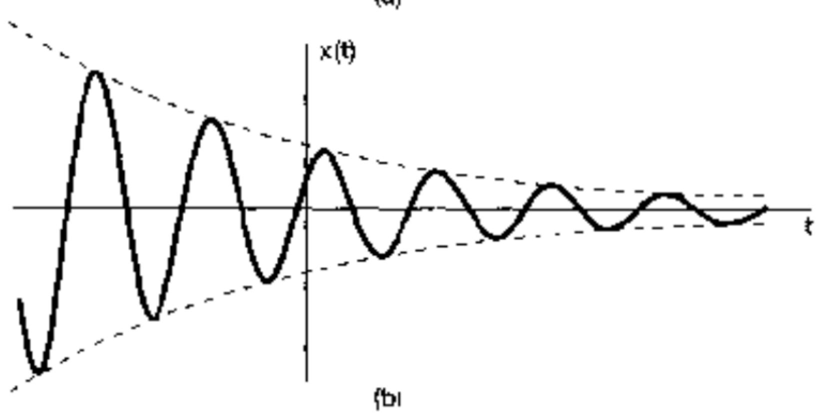
$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

General Complex Exponential (cont.)

- Thus for r=0, the real and imaginary parts of a complex exponential are sinusoidal.
- For r>0 they correspond to sinusoidal signals multiplied by a growing exponential.
- For r < 0, they correspond to sinusoidal signals multiplied by a decaying exponential.
- As shown below: (a) is growing sinusoidal signal when r>0, (b) is decaying sinusoid when r<0.



(a)



General Complex Exponential (cont.)

- Sinusoidal signals multiplied by decaying exponentials are commonly referred to as damped signals.

Discrete-Time Complex Exponential

- A discrete-time complex exponential function has the form:

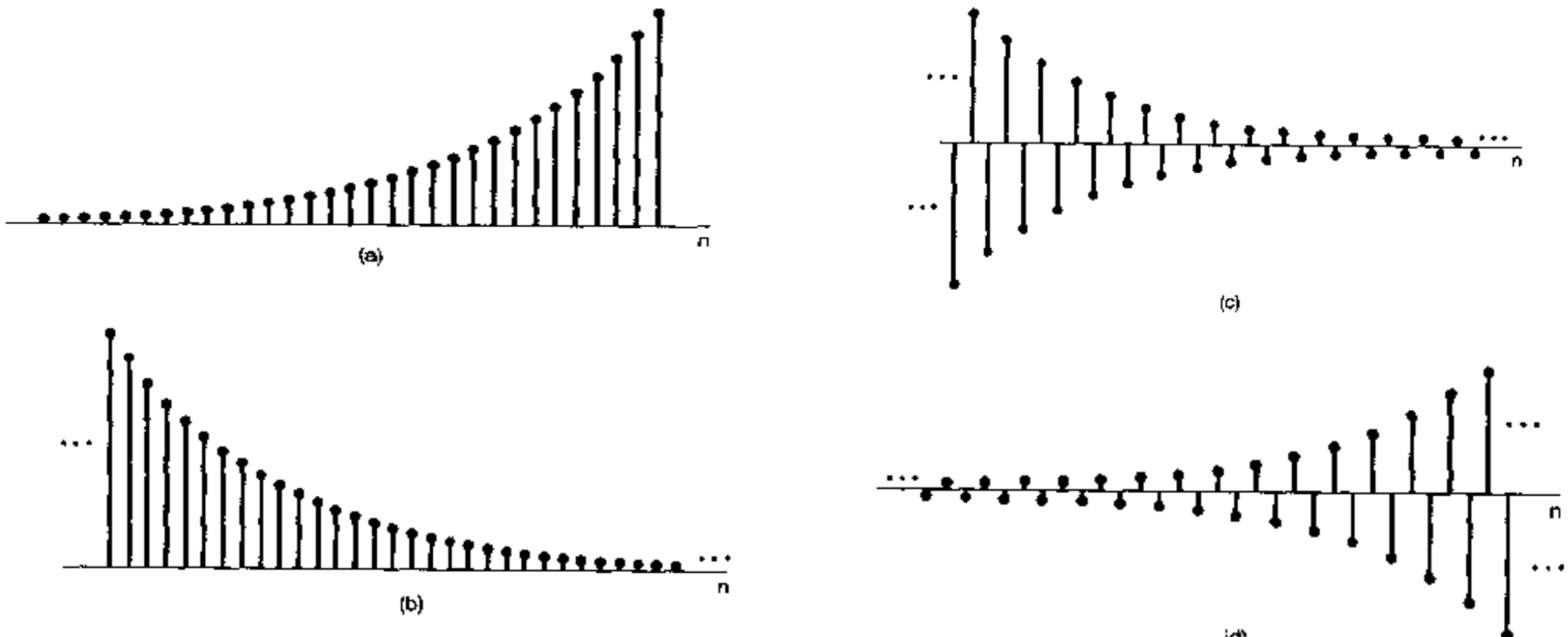
$$x[n] = Ce^{\beta n}$$

- Where C, β belongs to Complex. Letting $\alpha = e^{\beta}$:

$$x[n] = C\alpha^{n}$$

Real-valued Complex Exponential

- x[n] is a real-valued complex exponential when C belongs to R and α belongs to R.
- In this case, $x[n]=C\alpha^n$ is a monotonic decreasing function when 0 < α <1 and is a monotonic increasing when α > 1.



The real exponential signal (a) α>1, (b) 0< α <1, (c) -1< α <0, (d) α < -1

Complex-valued Complex Exponential

- x[n] is a complex-valued complex exponential when C,α belongs to complex.
- In this case C and α can be written as:

$$C = |C| e^{j\theta} \quad and \quad \alpha = |\alpha| e^{j\Omega_0}$$

$$Comsequently,$$

$$x[n] = C\alpha^n = |C| e^{j\theta} \left(|\alpha| e^{j\Omega_0} \right)^n$$

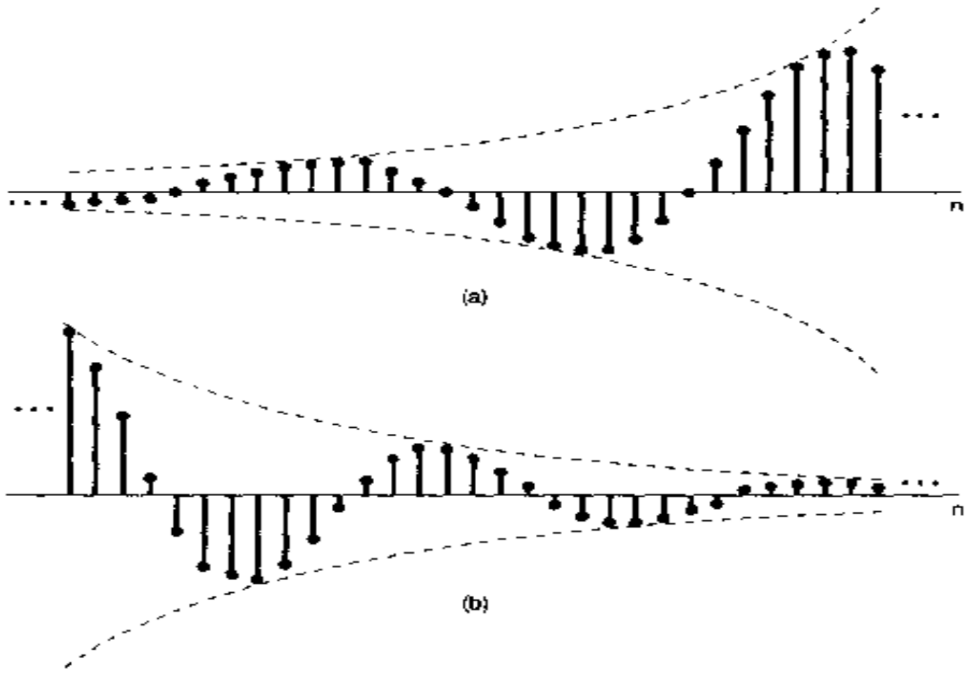
$$= |C||\alpha|^n e^{j(\Omega_0 n + \theta)}$$

$$= |C||\alpha|^n \cos(\Omega_0 n + \theta) + j|C||\alpha|^n \sin(\Omega_0 n + \theta)$$

Complex-valued Complex Exponential (cont.)

- Three cases can be considered here:
  - When $|\alpha|=1$, then $x[n] = |C|\cos(\Omega_0 n+\theta) + j\,|C|\sin(\Omega_0 n+\theta)$ and it has sinusoidal real and imaginary parts (not necessarily periodic though).
  - When $|\alpha| > 1$, then $|\alpha|^n$ is a growing exponential, so the real and imaginary parts of x[n] are the product of this with sinusoids.
  - When $|\alpha| < 1$, then the real and imaginary parts of x[n] are sinusoids sealed by a decaying exponential.

(a) Growing Discrete-time sinusoidal signals (b) decaying discrete time sinusoid



Periodic Complex Exponential

- Consider $x[n] = Ce^{j\Omega_0 n}, \Omega_0 \in R$. We want to study the condition for x[n] to be periodic.
- The periodicity condition requires that, for some N>0,

$$x[n+N] = x[n], \quad \forall n \in Z$$

- Since $x[n] = Ce^{j\Omega_0 n}$, it holds that:

$$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} e^{j\Omega_0 N} = e^{j\Omega_0 n}, \quad \forall n \in Z$$

- This is equivalent to:

$$e^{j\Omega_0 N} = 1 \quad or \quad \Omega_0 N = 2\pi m, \quad for \quad some \quad m \in Z$$

- Therefore, the condition for periodicity of x[n] is: $\Omega_0 = \frac{2\pi m}{N}$
  - For some m belongs to Z and some N>0, N belongs to Z.

Periodic Complex Exponential (cont.)

- Thus $x[n] = e^{j\Omega o n}$ is periodic if and only if Ω_0 is a rational multiple of 2π.
- The fundamental period is:

$$N = \frac{2\pi m}{\Omega_0}$$

- Where we assume that m and N are relatively prime, gcd (m,n) =1, i.e., m/N is in reduced form.

Impulse & Step Functions

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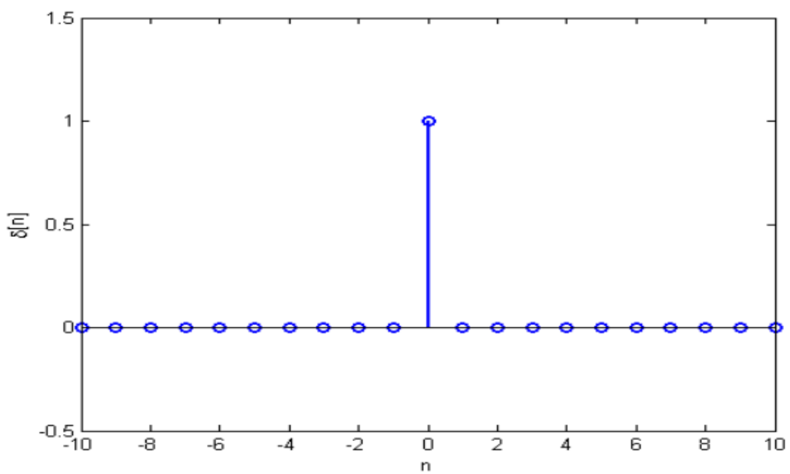
Discrete-time Impulse & Step Functions

- The discrete-time unit impulse signal δ[n] is defined as:

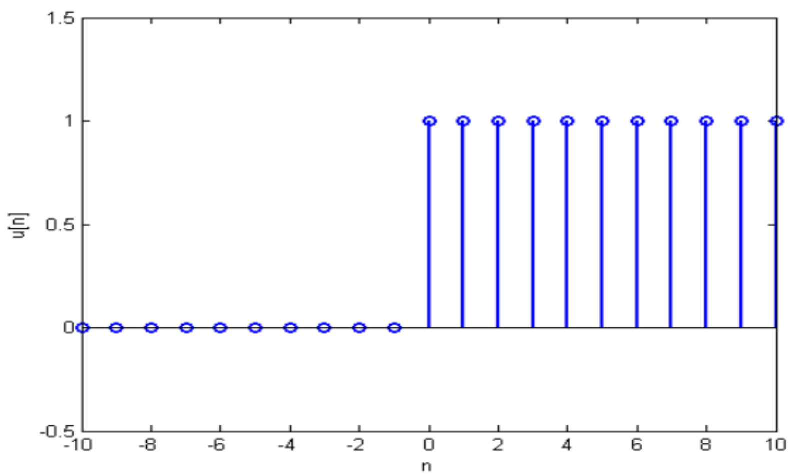
$$\delta(n) = \begin{cases} 1 & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases}$$

- The discrete-time unit step signal u[n] is defined as:

$$u(n) = \begin{cases} 1 & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$



(a) $\delta[n]$



(b) $u[n]$

Relationship B/w Unit Impulse & Unit Step Sequences

- Discrete time unit impulse is the first difference of the discrete time unit step. I.e.; $\delta[n]=u[n]-u[n-1]$
- Discrete time unit step is the running sum of the discrete time unit impulse or unit sample. i.e.;

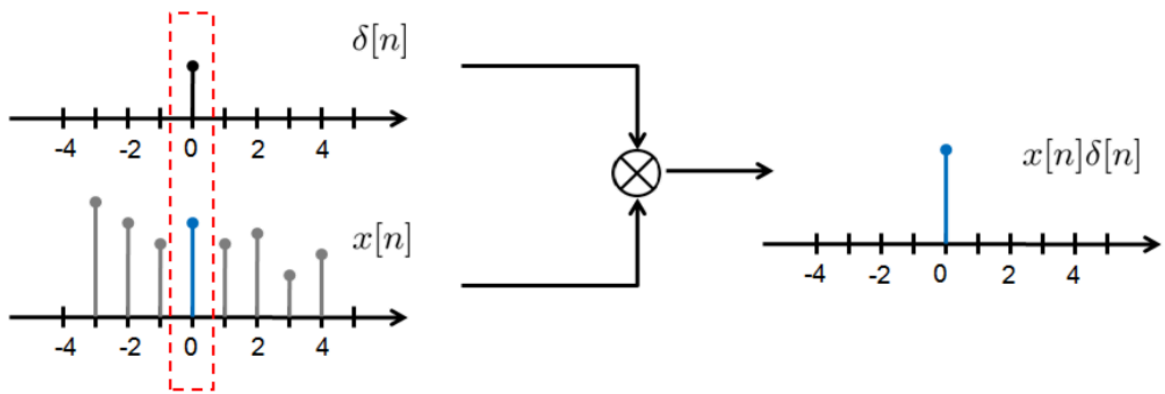
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Property of δ[n]

- **Sampling Property:**
- By the definition δ[n], $\delta[n-n_0] = 1$ if $n=n_0$ and 0 otherwise.
- Therefore,

$$x[n]\delta[n-n_0] = \begin{cases} x[n], & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$
$$= x[n_0]\delta[n-n_0]$$

- As a special case when $n_0=0$, we have x[n] δ[n]=x[0] δ[n].
- When a signal x[n] is multiplied with δ[n], the output is a unit impulse with amplitude x[0].



Property of δ[n]

- **Shifting Property:**
- Since x[n] δ[n]= x[0] δ[n] and $\sum_{n=-\infty}^{\infty} \delta[n] = 1$, we have

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n] = \sum_{n=-\infty}^{\infty} x[0]\delta[n] = x[0]\sum_{n=-\infty}^{\infty} \delta[n] = x[0]$$

- And similarly:

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = \sum_{n=-\infty}^{\infty} x[n_0]\delta[n-n_0] = x[n_0]$$

- In general, the following result holds:

$$\sum_{n=a}^{b} x[n]\delta[n-n_0] = \begin{cases} x[n_0], & if \quad n_0 \in [a,b] \\ 0, & if \quad n_0 \notin [a,b] \end{cases}$$

Continuous-time Impulse & Step Functions

- The Dirac delta is defined as:

$$\delta(t) = \begin{cases} 1 & for \quad t = 0 \\ 0 & for \quad t \neq 0 \end{cases}$$

- Where:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- The unit step function is defined as:

$$u(t) = \begin{cases} 1 & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

Property of δ[t]

- The properties of δ(t) are analogous to the discrete-time case:
- **Sampling Property:**

$$x(t)\delta(t) = x(0)\delta(t)$$

- Note that x(t) δ(t) = x(0) when t=0 and x(t) δ(t) =0 when t≠0.
- Similarly we have:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$for \quad any \quad t_0 \in R$$

Property of δ[n]

- **Shifting Property:**
- The shifting property follows from the sampling property.
- Integrating x(t) δ(t) yields:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = \int_{-\infty}^{\infty} x(0)\delta(t)\,dt = x(0)\int_{-\infty}^{\infty} \delta(t)\,dt = x(0)$$

- Similarly, one can show that:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

Energy & Power Signals

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Discrete-time Energy & Power Signals

- The energy of discrete time signal x(n) can be represented by:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- If the energy of the signal x(n) is finite i-e.; $0<E<\infty$, the signal is called an energy signal.
- The average power P of discrete time signal is represented by:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \left\{ \sum_{n=-N}^{N} |x(n)|^2 \right\}$$

- The signal x(n) is said to be power signal if $0<P<\infty$.

Continuous-time Energy & Power Signals

- A signal with finite signal energy is called an energy signal.
- A signal with infinite signal energy and finite average signal power is called a power signal.
- In signal processing, total energy of signal x(t) is defined as:

$$E(t) = \lim_{L\to\infty} \int_{-L}^{L} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

  - Where |x(t)| denotes the magnitude of x(t)
  - Energy of a signal is defined as a sum of square of magnitude.
- The average power of a signal is defined as:

$$P(t) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |x(t)|^2\, dt = \lim_{L\to\infty} \frac{E(t)}{2L}$$

- If x(t) is periodic, then its average power becomes:

$$P = \frac{1}{T} \int_{0}^{T} |x(t)|^2\, dt$$

Continuous-time Energy & Power Signals (cont.)

- An energy signal has finite energy, $0 < E < \infty$ and P=0.
- I-e; energy signals have values only in limited time duration.
- Power signal is not limited in time.
- It always exist from beginning to end and it never ends.
- For example, sine wave in infinite length is power signal.
- The energy of power signal is infinite but the power of the power signal is finite,$0 < P < \infty$ and $E=\infty$.
- A signal can be an energy signal, power signal or neither type.
- A signal can not be both an energy signal and a power signal.

Example #1

- Find whether the following signals are energy signal, power signal or neither of them:
  - $x(n)=(-0.3)^n u(n)$
  - $x(n)=2u(n)$
  - $y(t) = A \sin(\omega_0 t + \theta)$
  - $y(t) = e^{-bt} u(t)$

Thankyou

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