Signal & Systems

Convolution

26TH October 16

What is Convolution?

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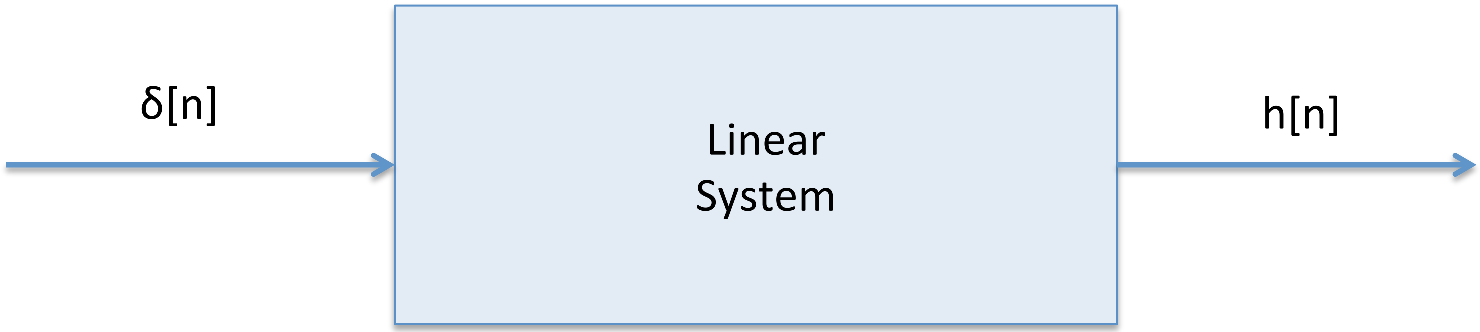
Introduction

- To study LTI systems, linear convolution plays an important role.
- Majority of discrete time systems in practice are shift invariant and linear and in many cases we take continuous systems as linear systems.
- The input signal can be decomposed into a set of impulses, each of which is scales and shifted delta/impulse function.
- The output from each impulse is scaled and shifted version of the impulse response.
- Then the overall output signal can be added to form one output.
- That is if we know the system's impulse response we can calculate the output for any possible input.
- This response is known as convolution kernel.

Introduction (cont.)



- Here $y[n] = x[n] * h[n]$.
- It is a formal mathematical operation i-e; *.
- In this we convolve two signals and produces the third signal.
- It is basically a relationship between three signals i-e; input signal, the impulse response and the output signal.

The Convolution Sum

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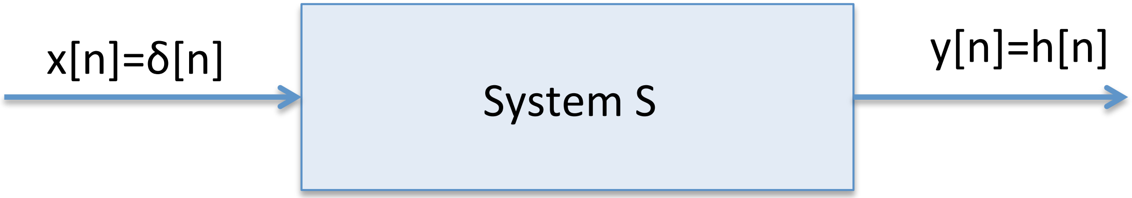
Representation of Discrete-Time Signals in Terms of Impulses

- Denote by h[n] the "impulse response" of an LTI system S.
- The impulse response is the response of the system to a unit impulse input.
- The definition of an unit impulse is: $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



- Let consider:

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

Representation of Discrete-Time Signals in Terms of Impulses (cont.)

- Using the above fact we get the following equalities:

$$x[n]\delta[n] = x[0]\delta[n], \quad (n_0 = 0)$$

$$x[n]\delta[n-1] = x[1]\delta[n-1], \quad (n_0 = 1)$$

$$x[n]\delta[n-2] = x[2]\delta[n-2], \quad (n_0 = 2)$$

$$\vdots \qquad\qquad \vdots$$

$$= x[n]\left(\sum_{k=-\infty}^{\infty} \delta[n-k]\right) \qquad = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The sum on the left hand side is:

$$x[n]\left(\sum_{k=-\infty}^{\infty} \delta[n-k]\right) = x[n]$$

- Because $\sum_{k=-\infty}^{\infty} \delta[n-k] = 1$ for all n.

Representation of Discrete-Time Signals in Terms of Impulses (cont.)

- The sum on the right hand side is:

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Therefore, equating the left hand side and right hand side yields:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Any signal x[n] can be expressed as a sum of impulses.
- Suppose we know that the impulse response of an LTI system is h[n] and we want to determine the output y[n].
- To do so we first express x[n] as a sum of impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Representation of Discrete-Time Signals in Terms of Impulses (cont.)

- For each impulse δ [n-k], we can determine its impulse response, because for an LTI system: $\delta[n-k] \rightarrow h[n-k]$
- Consequently, we have:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n]$$

- Where the equation: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is known as the convolution equation.

Evaluating Convolution

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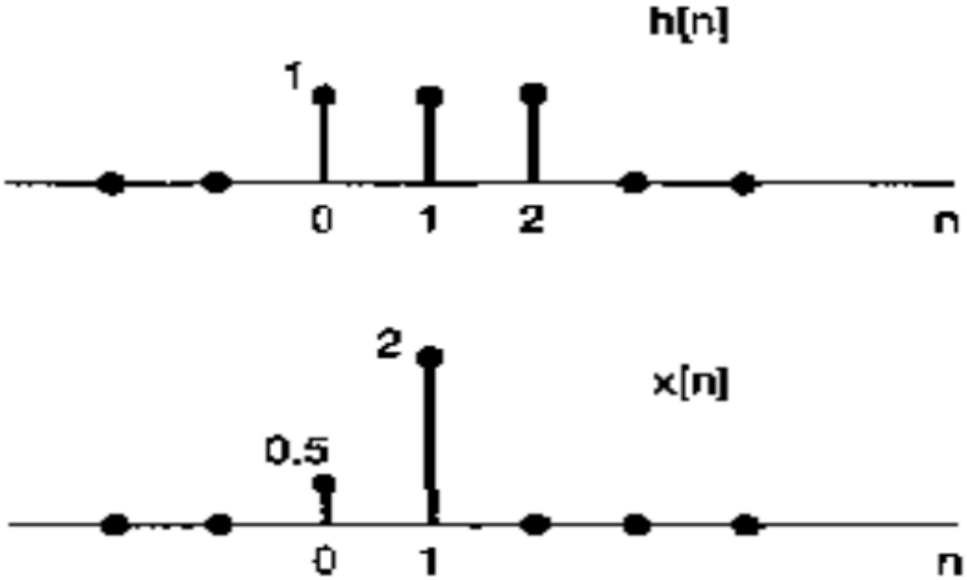
How to Evaluate Convolution?

- There are three basic steps used to evaluate convolution of any signal:
  - Flip
  - Shift
  - Multiply and Add.

Example #1

❖ Consider an LTI system with impulse response h[n] and input x[n] as illustrated below:



❖ For this case since only x[0] and x[1] are nonzero, can be expressed as follows:

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$

Example #2

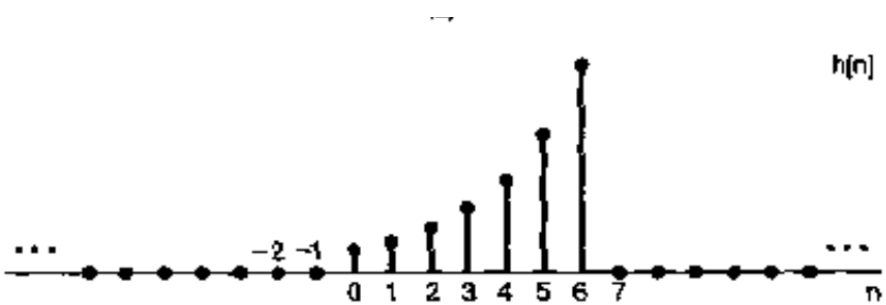
❖ Consider the signal x[n] and the impulse response h[n] shown below:

Example #3

- Consider the two sequences:

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & otherwise \end{cases} \qquad h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & otherwise \end{cases}$$

- These signals are depicted below for a positive value of α > 1.
- Inorder to calculate the convolution of the two signals, it is convenient to consider five separate intervals for n.



The Convolution Integral

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Continuous-Time Convolution

- The continuous-time case, is analogous to the discrete time case.
- In continuous time signal the signal decomposition is:

$$y(t) = T[x(t)] = T\left[\sum_{n=-\infty}^{\infty} x(\tau)\delta(n-\tau)\right]$$

- The continuous time convolution is defined as:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = h(t) * x(t)$$

Convolution Theorem

- h(t) is the impulse response of an analog or continuous time system for an input δ(t).
- Hence for a Linear system its output for a shifted impulse δ(t-T) is h(t-T).
- Therefore,

$$\delta(t) \rightarrow h(t)$$

$$\delta(t-T) \rightarrow h(t-T)$$

$$x(t)\delta(t-T) \rightarrow x(T)h(t-T)$$

- Obviously,

$$\int_{-\infty}^{\infty} x(T)\delta(t-T)\,dT \rightarrow \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$$

Convolution Theorem (cont.)

- Therefore, we can write by the linearity of the system:

$$y(t) = \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$$

- This is the convolution of x(t) with h(t).

$$\therefore\ y(t) = \quad Convolution \quad of \quad h(t) \quad \& \quad x(t)$$

$$= x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(T)h(t-T)\,dT$$

Example #4

- Consider the signal x(t) = e^{-at} u(t) for a > 0, and impulse response h(t) = u(t). The output y(t) is:
- Case A: t > 0;

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) u(t-\tau) d\tau$$

$$= \int_{0}^{t} e^{-a\tau}\, d\tau = \frac{1}{-a}\left[1-e^{-at}\right]$$

- Case B: t ≤ 0;

$$y(t) = 0$$

- Therefore,

$$y(t) = \frac{1}{a}\left[1-e^{-at}\right] u(t)$$

Evaluating Continuous Convolution

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Steps for Graphical Convolution

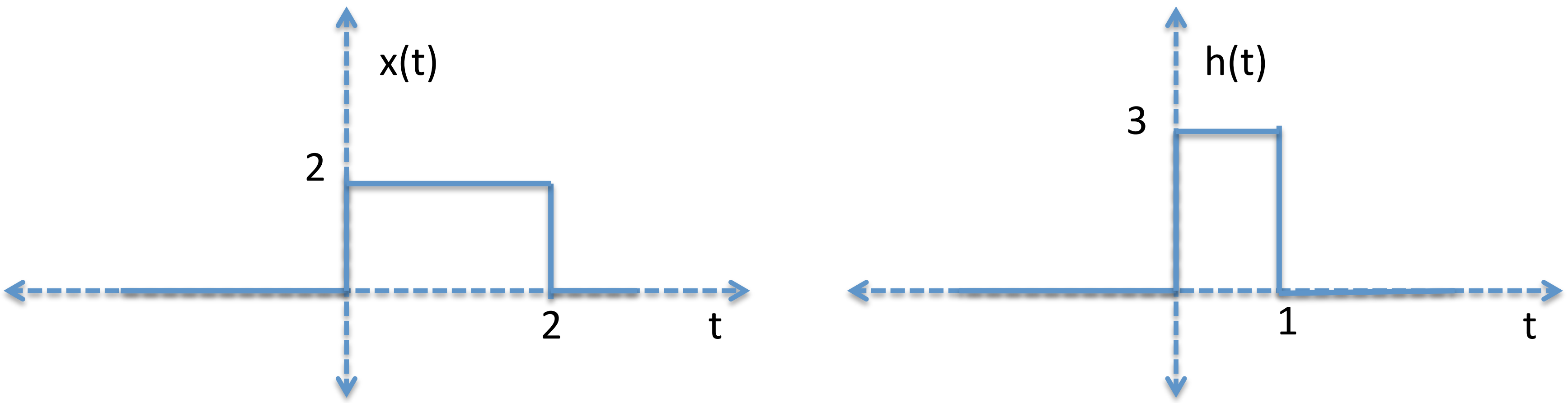
- Steps for Graphical convolution are:
  - Re-write the signals as functions of τ: $x(\tau)$ & $h(\tau)$.
  - Flip one of the signals around t=0 to get either $x(-\tau)$ or $h(-\tau)$.
    - It is best to flip the signal with shorter duration.
    - We'll flip $h(\tau)$ to get $h(-\tau)$ here, for notational purposes.
  - Find edges of the flipped signal.
    - Find the left-hand edge of τ value of $h(-\tau)$: call it $\tau_{L,o}$
    - Find the right hand edge τ value of $h(-\tau)$: call it $\tau_{R,o}$
  - Shift $h(-\tau)$ by an arbitrary value of t to get $h(t-\tau)$ and get its edges.
    - Find the left-hand edge of τ value of $h(t-\tau)$ as a function t: call it $\tau_{L,t}$ it will always be $\tau_{L,t}=t+\tau_{L,t}$
    - Find the right hand edge τ value of $h(t-\tau)$ as a function t: call it $\tau_{R,t}$ it will always be $\tau_{R,t}=t+\tau_{R,t}$

Steps for Graphical Convolution (cont.)

- Find regions of τ-Overlap:
  - Find intervals of t over which the product $x(\tau)h(t-\tau)$ has a single mathematical form in terms of τ.
  - In each region find interval of t that makes the identified overlap happen.
- For each region form the product of $x(\tau)h(t-\tau)$ and integrate.
- Assemble the output from the output time-sections for all the regions.

Example #5

❖ Graphically convolve two signals:



Properties of Linear Time-Invariant Systems

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Properties

- The convolution properties are as follows:
  - Commutative Property
  - Associative Property
  - Distributive Property

Commutative Property

- A basic property of convolution in both continuous and discrete time is that it is a commutative operation.
- In discrete time it is: $$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty} h[k]x[n-k]$$
- In continuous time it is:

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Proof: In the discrete time case if we let r=n-k or equivalently k=n-r then:

$$x[n]*h[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n-k]=\sum_{r=-\infty}^{+\infty} x[n-r]h[r]=h[n]*x[n]$$

- With this substitution of variables, the roles of x[n] and h[n] are interchanged.

Commutative Property (cont.)

- This property states that one of the two forms for computing convolutions in discrete time and continuous time may be easier to visualize, but both forms always result in the same answer.

Distributive Property

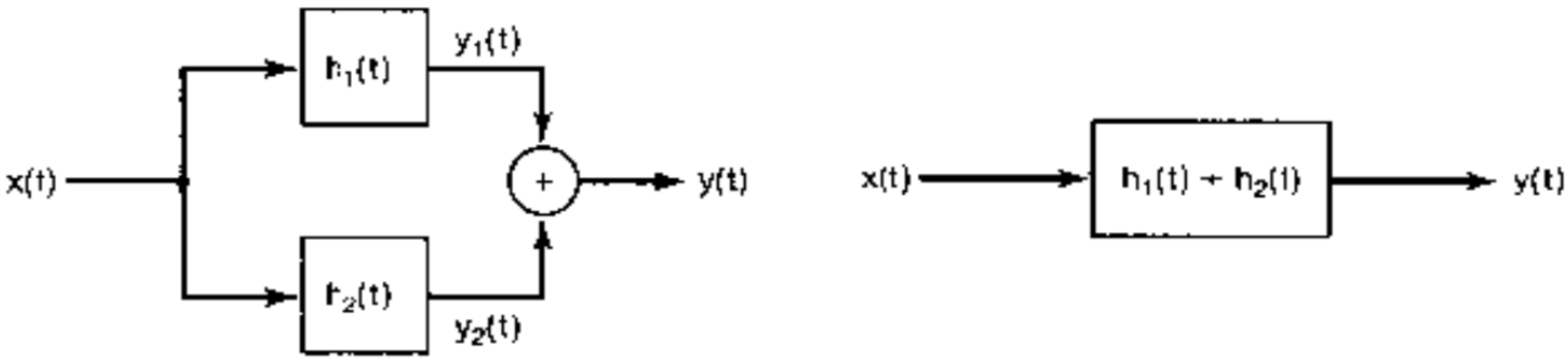
❖ Convolution distributes over addition so that in discrete time:

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

❖ In continuous time:

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$

❖ Interpretation of Distributive property of convolution for a parallel interconnection of LTI systems is shown below:



Distributive Property (cont.)

- The two systems with impulse responses $h_1(t)$ and $h_2(t)$ have identical inputs and their outputs are added.
- Since:

$$y_1(t) = x(t) * h_1(t)$$

$$and$$

$$y_2(t) = x(t) * h_2(t)$$

- The system of above figure has output:

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

- The system of second figure has the output:

$$y(t) = x(t) * \left[h_1(t) + h_2(t)\right]$$

- Comparing both the above results we see that the systems in above figures are identical.
- In same way distributive property of discrete time can also be proved.

Associative Property

- Another important and useful property of convolution is associative property.
- In discrete time: $x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$
- In continuous time: $x(t)*[h_1(t)*h_2(t)] = [x(t)*h_1(t)]*h_2(t)$
- As a consequence of the associative property, the expressions:

$$y[n] = x[n]*h_1[n]*h_2[n]$$

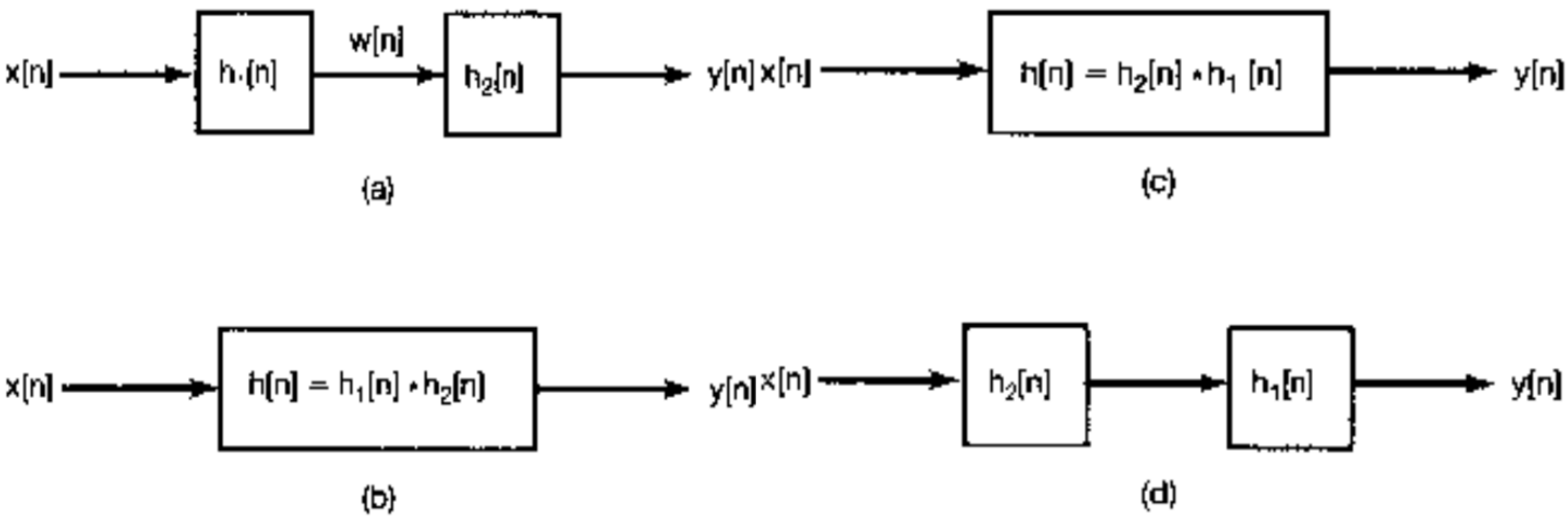
$$and$$

$$y(t) = x(t)*h_1(t)*h_2(t)$$

- Are unambiguous. That is it does not matter in which order we convolve these signals.

Associative Property (cont.)

- An interpretation of the associative property is illustrated for discrete time systems in figures below:



$$y[n]=w[n]*h_2[n]=\left(x[n]*h_1[n]\right)*h_2[n], \quad fig(a)$$

$$y[n]=x[n]*h[n]=x[n]*\left(h_1[n]*h_2[n]\right), \quad fig(b)$$

Associative Property (cont.)

- According to the associative property the series interconnection of the two systems in fig(a) is equivalent to the single system in fig(b).
- This can be generalized to an arbitrary number of LTI systems in cascade and the analogous interpretation and conclusion also hold in continuous time.
- By using the commutative property together with the associative property, we find another very important property of LTI systems.
- From fig(a) and (b) we can conclude that the impulse response of the cascade of two LTI system is the convolution of their individual impulse responses.
- Since convolution is commutative we can compute this convolution of $h_1[n]$ and $h_2[n]$ in either order.

Thankyou

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