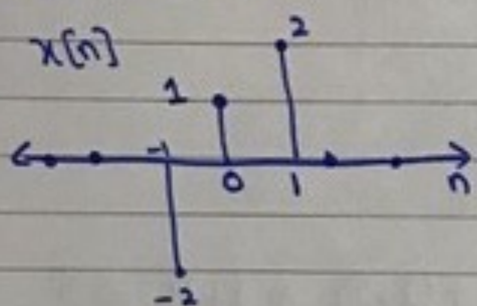
Lecture Notes
31st October 2016

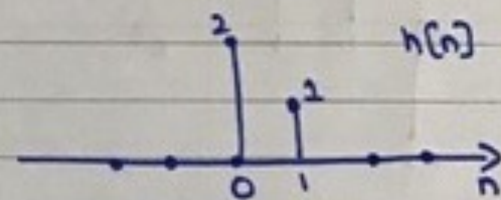
Date MONDAY | 31st Oct 2016 LECTURE #5

CONVOLUTION EXAMPLES:-

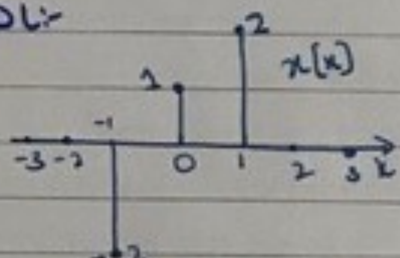
EXAMPLES #1:-



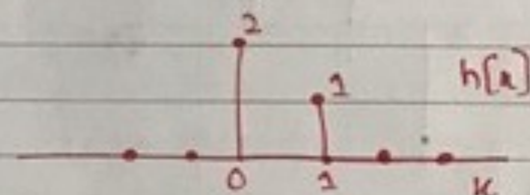
$*$

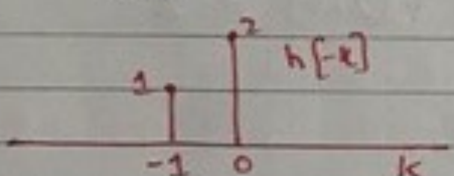


SOL:-

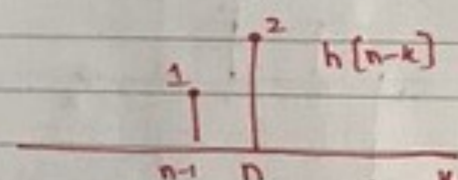


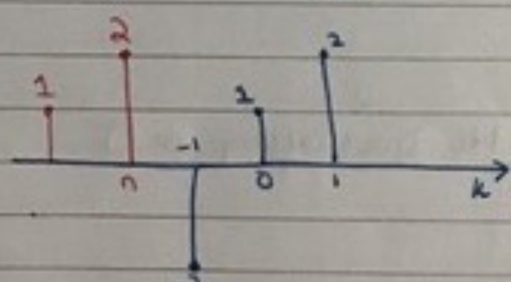
$*$



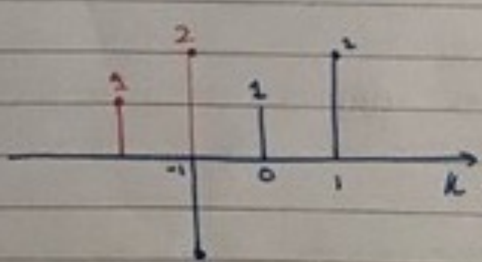


$\xrightarrow{h[n-k]}$

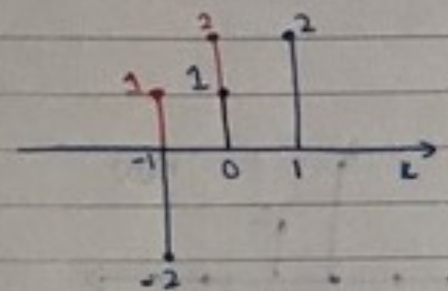




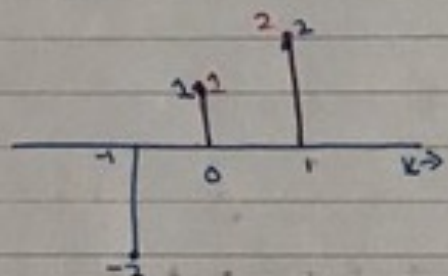
$n < -1$; $\sum_{n=-\infty}^{\infty} x[k]\, h[n-k]$ = No overlapping ⇒ 0



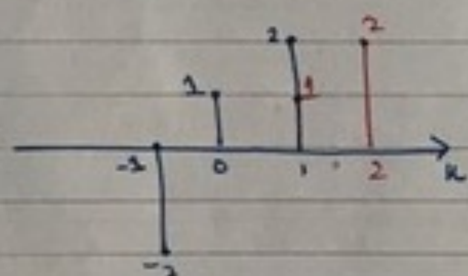
$y[n] = 2 \times (-2) \Rightarrow -4$



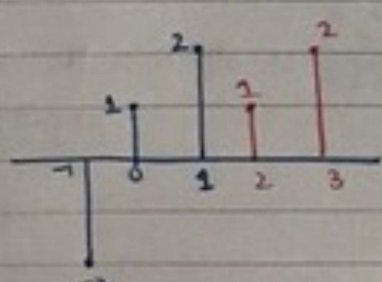
$y[0] = 1\times(-2) + (2\times1) \Rightarrow -2+2 \Rightarrow 0$



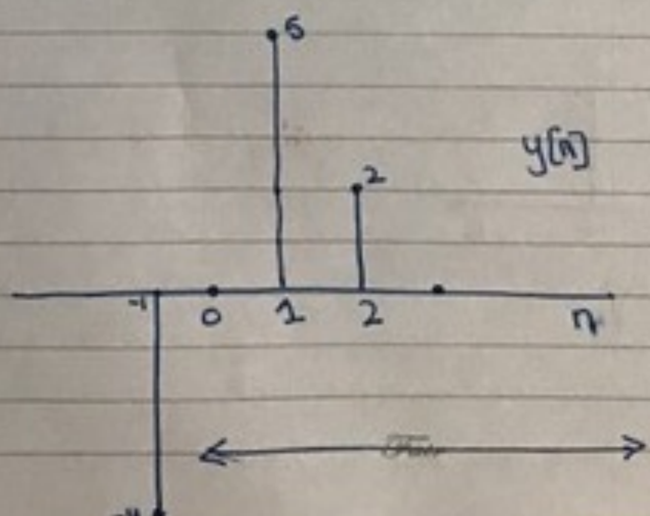
$y[1] = (1\times1) + (2\times2) \Rightarrow 1+4 \Rightarrow 5$



$y[2] = (2\times1) + (2\times0) \Rightarrow 2$

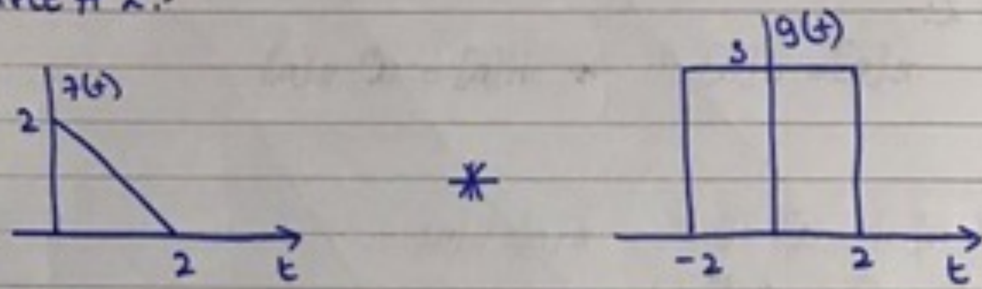


$y[3] \Rightarrow$ No overlapping $\Rightarrow 0$



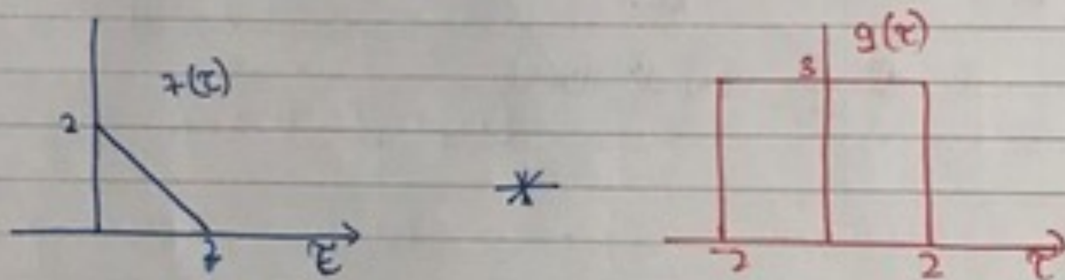
Day/Date

EXAMPLE # 2:-

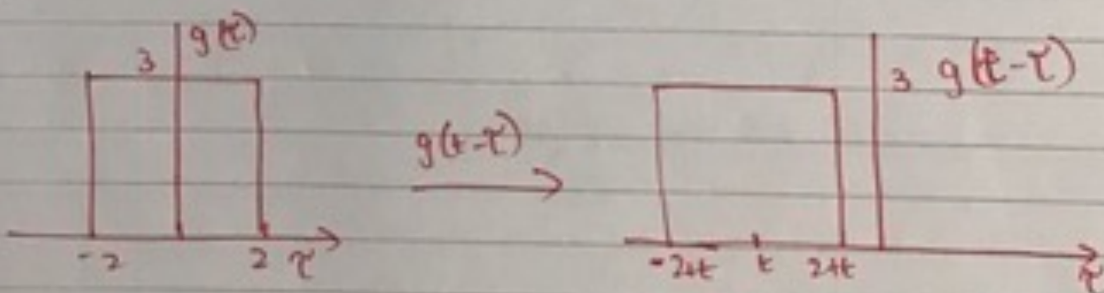


SOL:-

Step #1 :- replace t with τ.

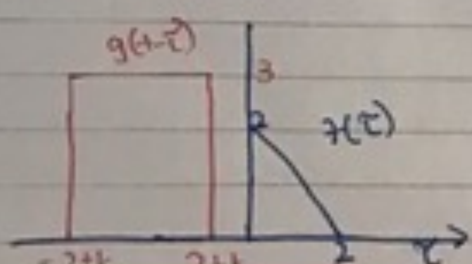


Step #2:- Flip & Shift one signal Here we are flipping and shifting g(τ).



Step # 3 :- Start overlapping.

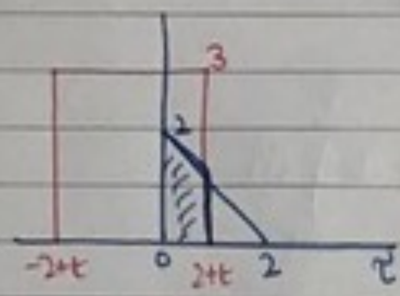
1 → $t < -2$



No overlapping so $y(t) = 0$

Date

2→ $-2 \le t < 0$



$$y(t) = \int_0^{2+t} 3(-\tau+2)\,d\tau$$

$$= -3\int_0^{2+t} \tau\,d\tau + 6\int_0^{2+t} d\tau$$

$$= -3\left[\frac{\tau^2}{2}\right]_0^{2+t} + 6\left[\tau\right]_0^{2+t}$$

$$= -3\left[\frac{(2+t)^2}{2} - 0\right] + 6\left[2+t-0\right]$$

$$= -3\left[\frac{4+4t+t^2}{2}\right] + 6(2+t)$$

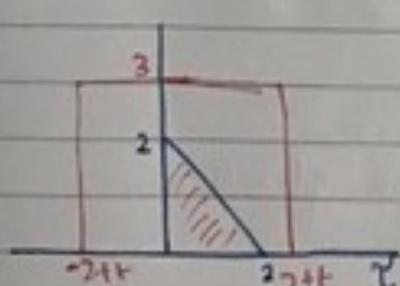
$$= \frac{-12-12t-3t^2}{2} + 12+6t$$

$$= \frac{-12-\cancel{12t}-3t^2+24+\cancel{12t}}{2}$$

$$= \frac{-3t^2+12}{2} \Rightarrow -\frac{3t^2}{2} + \frac{\cancel{12}^{6}}{\cancel{2}}$$

$$y(t) = \frac{-3t^2}{2} + 6$$

3→ $0 \le t < 2$



$$y(t) = \int_0^{2} 3(-\tau+2)\,d\tau$$

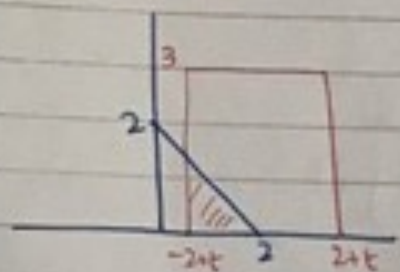
$$= -3\int_0^{2} \tau\,d\tau + 6\int_0^{2} d\tau$$

$$= -3\left[\frac{\tau^2}{2}\right]_0^{2} + 6\left[\tau\right]_0^{2}$$

$$y(t) = -3\left[\frac{\cancel{4}^2}{\cancel{2}_1}\right] + 6\left[2\right] = -6+12 \Rightarrow 6$$

Day/Date

4 → $2 \le t < 4$



$$y(t) = \int_{-2+t}^{2} 3(-\tau+2)\,d\tau$$

$$= -3\int_{-2+t}^{2} \tau\,d\tau + 6\int_{-2+t}^{2} d\tau$$

$$= -3\left[\frac{\tau^2}{2}\right]_{-2+t}^{2} + 6\left[\tau\right]_{-2+t}^{2}$$

$$= -3\left[\frac{4}{2} - \frac{(-2+t)^2}{2}\right] + 6\left[2+2-t\right]$$

$$= -3\left[\frac{\cancel{4} - t^2 + 4t - \cancel{4}}{2}\right] + 6\left[4-t\right]$$

$$= -3\left[\frac{-t^2+4t}{2}\right] + 24 - 6t$$

$$= \frac{3t^2}{2} - \frac{\overset{6}{\cancel{12}}t}{\cancel{2}} + 24 - 6t$$

$$= \frac{3t^2}{2} - 6t + 24 - 6t$$

$$= \frac{3t^2}{2} + 24 - 6t \text{ Ans}$$

$$-3\left[\frac{4^2}{\cancel{2}} - \frac{(t-2)^2}{2}\right] + 6\left[\cancel{2} - (-2+t)\right]$$

$$= -3\left[\cancel{2} - \frac{t^2 - 4t + 4}{2}\right] + 6\left[2+2-t\right]$$

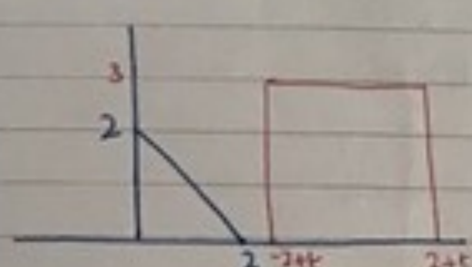
$$= -6 + \frac{3t^2 - 12t + 12}{2} + 6\left[4-t\right]$$

$$= -6 + \frac{3t^2 - 12t + 12}{2} + 24 - 6t$$

$$= \frac{\cancel{12} + 3t^2 - \cancel{12t} + \cancel{12} + 48 - \cancel{12t}}{2}$$

$$= \frac{3t^2 + 48}{2} \Rightarrow \frac{3t^2}{2} + 24$$

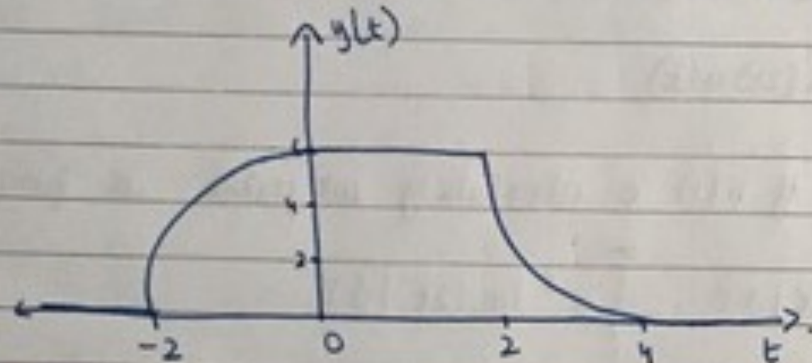
5 → $t \ge 4$



No overlapping so $y(t) = 0$

Day/Date

$$y(t) = f(t) * g(t) = \begin{cases} 0 & \text{for } t < -2 \\ -\frac{3t^2}{2} + 6 & \text{for } -2 \le t < 0 \\ 6 & \text{for } 0 \le t < 2 \\ \frac{3t^2}{2} - 12t + 24 & \text{for } 2 \le t < 4 \\ 0 & \text{for } t \ge 4 \end{cases}$$



EXAMPLE #3:-

$$x(t) = e^{-at} u(t) \quad * \quad h(t) = e^{-bt} u(t), \quad b \neq a$$

SOL:-

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-a\tau} e^{-bt} e^{+b\tau} u(\tau) u(t-\tau)\, d\tau$$

$$= e^{-bt} \int_0^t e^{-a\tau} e^{b\tau}\, d\tau \quad , \quad t > 0$$

$$= e^{-bt} \int_0^t e^{b\tau - a\tau}\, d\tau = e^{-bt} \left[\frac{e^{(-a+b)\tau}}{-a+b} \right]_0^t$$

$$= \cancel{e^{-bt} \left[\frac{e^{(-a+b)t}}{-a+b} - 0 \right] = e^{-bt} \left[e^{-at} e^{bt} \right.}$$

$$= e^{-bt} \left[\frac{e^{(-a+b)t}}{-a+b} - \frac{e^{0}}{-a+b} \right] = e^{-bt} \left[\frac{e^{-at} e^{bt}}{-a+b} - \frac{1}{-a+b} \right]$$

$$= \frac{e^{-at} e^{-bt+bt} - e^{-bt}}{-a+b} \Rightarrow \frac{e^{-at} - e^{-bt}}{b-a} u(t)$$

Day/Date

when $t < 0$ then $y(t) = 0$.

EXERCISE PROBLEMS:-

PROBLEM #1:-

Which of the following impulse responses correspond(s) to stable LTI systems?

a) $h(t) = e^{-t}\cos(2t)u(t)$

SOL:-

We determine if $h(t)$ is absolutely integrable as follows:-

$$\int_{-\infty}^{\infty} |h(t)|\, d\tau = \int_{0}^{\infty} e^{-t}\,|\cos(2t)|\, d\tau$$

This integrable is clearly finite valued because $e^{t}|\cos(2t)|$ is an exponentially decaying function in the range $0 \le t < \infty$. Therefore $h(t)$ is the impulse response of the stable LTI system.

$\longleftrightarrow$

b) $h[n] = 3^n u[-n+10]$

SOL:-

We have to determine if $h[n]$ is absolutely summable as follows:-

$$\sum_{-\infty}^{\infty} |h[k]| = \sum_{-\infty}^{10} 3^k \approx \frac{3^n}{2}$$

Therefore $h[n]$ is the impulse response of a stable LTI system.

Day/Date

PROBLEM #2:-

Determine whether each system is causal and/or stable:-

a) $h[n] = 5^n u[3-n]$

Sol:-

$\sum_{n=-\infty}^{3} 5^n \Rightarrow \frac{625}{4} < \infty$ so it is stable.

But not causal as $h[n] \neq 0$ for $n < 0$.

b) $h(t) = e^{-4t} u(t-2)$

Sol:-

It is causal because $h(t) = 0$ for $t < 0$.

$\int_{-\infty}^{\infty} e^{-4t} u(t-2)\, dt \Rightarrow \frac{e^{-8}}{4} < \infty$ so it is stable.