Signal & Systems

Convolution Properties

31st October 16

Convolution Examples

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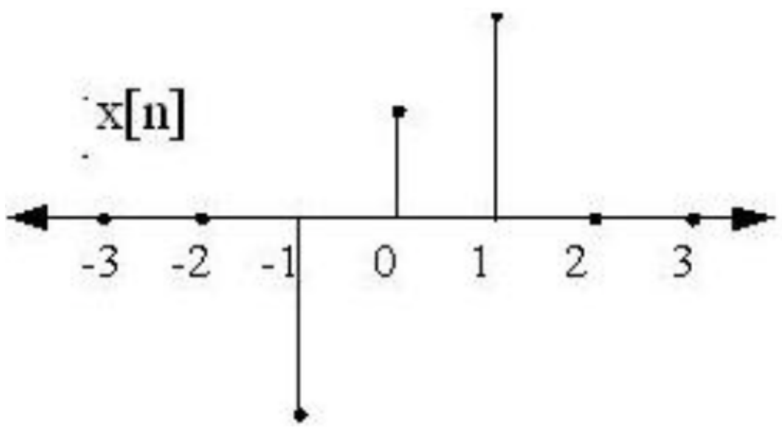
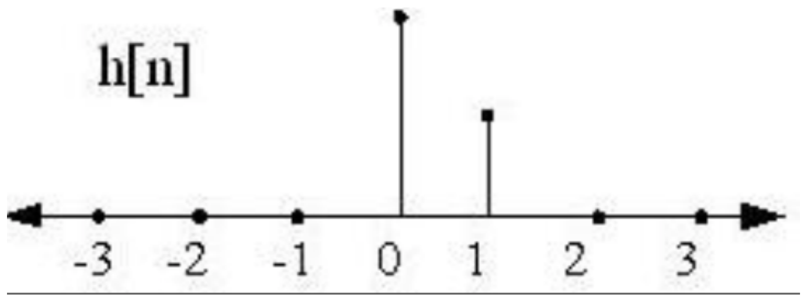
Example #1

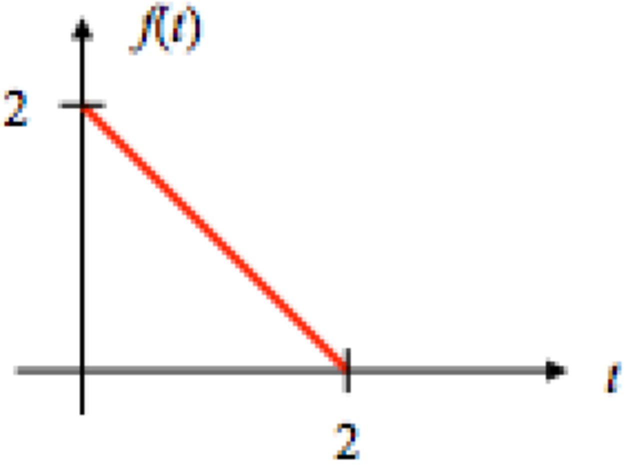
❖ Compute and plot y[n] = x[n] * h[n], where:

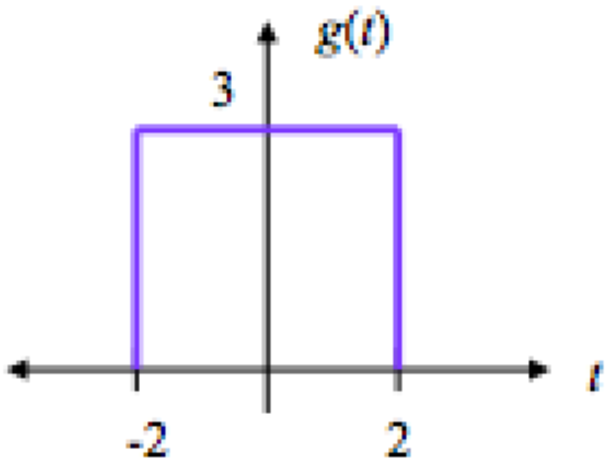




Example #2

- Convolve the following two functions:





Example #3

- Convolve the two signals given below:

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t), \quad b \neq a$$

System Properties & Impulse Response

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LTI Systems With & Without Memory

- A system is memoryless if the output depends on the current input only.
- An LTI system is memoryless if and only if h[n] = 0 for n≠0.
- In this case the impulse response has the form: h[n] = K δ[n] where K = h[0] is a constant and the convolution sum reduces to the relation:

$$y[n] = Kx[n]$$

- If a discrete time LTI system has an impulse response h[n] that is not identically zero for n≠0, then the system has memory.
- A continuous time LTI system is memoryless if h(t) = 0 for t ≠ 0, and such a memoryless LTI system has the form: y(t) =K x(t) for some constant K and has the impulse response:

$$h(t) = K\delta(t)$$

LTI Systems With & Without Memory (cont.)

- Note that if K=1 in discrete and continuous time impulse response then these systems become identity systems with output equal to the input and with unit impulse response equal to the unit impulse.
- In this case the convolution sum and integral formulas imply that:

$$x[n] = x[n] * \delta[n]$$

and

$$x(t) = x(t) * \delta(t)$$

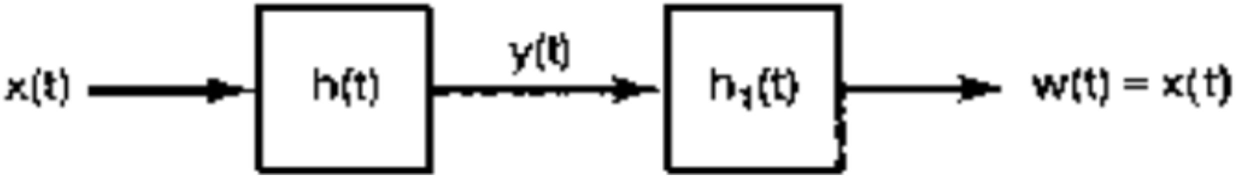
- Which reduces to the sifting properties of the discrete and continuous time unit impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

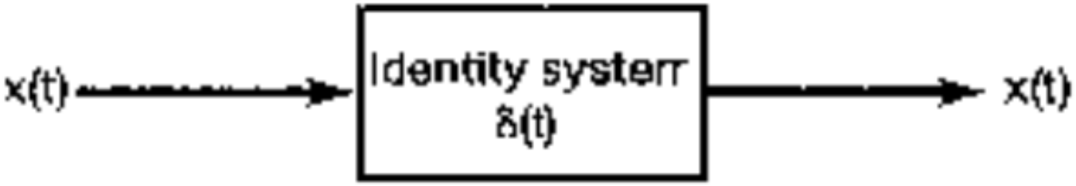
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

Invertibility of LTI Systems

- A system is invertible if and only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.
- If an LTI system is invertible, then it has an LTI inverse.
- Suppose we have a system with impulse response h(t). The inverse system with impulse response $h_1(t)$ results in w(t) = x(t) as shown below in series interconnection.



- This system is identical to the identity system shown below:



Invertibility of LTI Systems (cont.)

- Since the overall impulse response shown in first figure is h(t) * h_1(t), we have the condition that h_1(t) must satisfy for it to be the impulse response of the inverse system, i.e.,

$$h(t) * h_1(t) = \delta(t)$$

- Similarly in discrete time the impulse response h_1[n] of the inverse system for an LTI system with impulse response h[n] must satisfy:

$$h[n] * h_1[n] = \delta[n]$$

Example #4

- Consider the LTI system consisting of a pure time shift:

$$y(t) = x(t - t_0)$$

- If $t_0 > 0$, then the output at time t equals the value of the input at the earlier time $t - t_0$.
- If $t_0 = 0$, the system is the identity system and thus us memoryless.
- For any other value of t_0, this system has memory, as it responds to the value of the input at a time other than the current time.
- The impulse response for the system can be obtained by taking the input equal to δ (t), i.e., $h(t) = \delta(t - t_0)$
- Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0)$$

- That is the convolution of a signal with a shifted impulse simply shifts the signal.

Example #4 (cont.)

- To recover the input from the output i.e., to invert the system, all that is required is to shift the output back.
- The system with this compensating time shift is then the inverse system. That is if we take:

$$h_1(t) = \delta(t + t_0)$$

- Then:

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

- Similarly, a pure time shift in discrete time has the unit impulse response $\delta[n-n_0]$, so that convolving a signal with a shifted impulse is the same as shifting the signal.
- The inverse of the LTI system with impulse response $\delta[n-n_0]$ is the LTI system that shifts the signal in the opposite direction by the same amount i.e., the LTI system with impulse response $\delta[n+n_0]$.

Causality for LTI Systems

- The output of a causal system depends only on the present and past values of the input to the system.
- Theorem: An LTI system is causal if and only if:

$$h[n] = 0, \quad for \quad all \quad n < 0$$

- Proof: If S is causal, then the output y[n] cannot depend on x[k] for k>n.
- From the convolution equation:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- We must have:

$$h[n-k] = 0, \quad for \quad k > n$$

Causality for LTI Systems (cont.)

- Or equivalently $h[n-k]=0, \quad for \quad n-k<0$
- Setting m = n − k, we see that: h[m] =0, for m < 0.
- Conversely, if h[k] = 0 for k < 0, then for input x[n],

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=0}^{\infty} h[k]x[n-k]$$

- Therefore, y[n] depends only upon x[m] for m ≤ n.

Stability for LTI Systems

- A system is stable if every bounded input produces a bounded output.
- Inorder to determine conditions under which LTI systems are stable, consider an input x[n] that is bounded in magnitude:

$$|x[n]| < B, \quad for \quad all \quad n$$

- Suppose that this input is applied to an LTI system with unit impulse response h[n].
- Then, using the convolution sum, we obtain an expression for the magnitude of the output:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

Stability for LTI Systems (cont.)

- Since the magnitude of the sum of a set of numbers is no longer than the sum of the magnitudes of the numbers, it follows from above equation that:

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

- As $|x[n-k]| < B$ for all values of k and n, then:

$$|y[n]| \le B \sum_{k=-\infty}^{\infty} |h[k]|, \quad for \quad all \quad n$$

- From above equation we can conclude that if the impulse response is absolutely summable that is, if:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- Then y[n] is bounded in magnitude and hence the system is stable.
- If above equation is not satisfied, there are bounded inputs that result in unbounded outputs.

Stability for LTI Systems (cont.)

- Thus in continuous time case the system is stable if the impulse response is absolutely integrable, i.e., if:

$$\int_{-\infty}^{\infty} \left| h(\tau) \right| d\tau < \infty$$

Example #5

- Consider a system that is a pure time shift in either continuous time or discrete time.
- Then in discrete time: $$\sum_{n=-\infty}^{\infty} \left|h[n]\right| = \sum_{n=-\infty}^{\infty} \left|\delta[n-n_0]\right| = 1$$
- While in continuous time: $$\int_{-\infty}^{\infty} h(\tau)\,d\tau = \int_{-\infty}^{\infty} \left|\delta(\tau - t_0)\right| d\tau = 1$$
- Thus we conclude that both of these systems are stable.
- Since if a signal is bounded in magnitude, so is any time-shifted version of that signal.
- Now if we consider the example of accumulator, the system is unstable as if we apply a constant input to an accumulator the output grows without bound.
- That is the system is unstable can also be seen from the fact that its impulse response u[n] is not absolutely summable:

Example #5 (cont.)

$$\sum_{n=-\infty}^{\infty} \left|u[n]\right| = \sum_{n=0}^{\infty} u[n] = \infty$$

- Similarly, consider the integrator, the continuous time counterpart of the accumulator:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

- This is an unstable system for precisely the same reason as that given for the accumulator, i.e., a constant input give rise to an output that grows without bound.
- The impulse response for the integrator can be found by letting x(t)=δ(t), in which case:

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

$$and$$

$$\int_{-\infty}^{\infty} \left|u(\tau)\right| d\tau = \int_{0}^{\infty} u(\tau)\, d\tau = \infty$$

Example #5 (cont.)

- Since the impulse response is not absolutely integrable, the system is not stable.

Thankyou

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