Lecture Notes
9th November 2016

WEDNESDAY / 9th NOV, 16 LECTURE: 7

EXAMPLE # 1:-

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$$

SOL:-

$T_1 \Rightarrow 1 \qquad T_2 = \frac{2\pi}{2\pi} \Rightarrow 1 \qquad T_3 = \frac{2\pi}{3\pi} \Rightarrow 2/3$

$T_1/T_2 \Rightarrow 1 \qquad T_1/T_3 = 1/_{2/3} \Rightarrow \frac{3}{2}$

$\because \cos t = \frac{e^{it} + e^{-it}}{2}$

$T = LCM\left(1, \frac{3}{2}\right) \Rightarrow 2$

$\because \sin t = \frac{e^{it} - e^{-it}}{2i}$

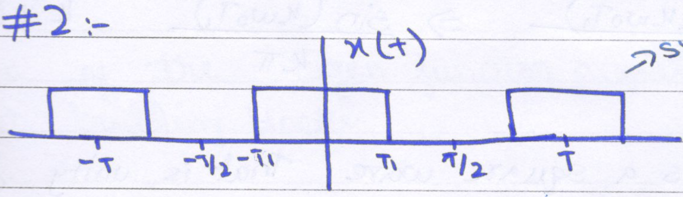
$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$$= 1 + \frac{1}{4}e^{j2\pi t} + \frac{1}{4}e^{-j2\pi t} + \frac{1}{2j}e^{j3\pi t} - \frac{1}{2j}e^{-j3\pi t}$$

$a_0 = 1$, $a_1 = a_{-1} = 0$, $a_2 = a_{-2} = \frac{1}{4}$, $a_3 = \frac{1}{2j}$, $a_{-3} = -\frac{1}{2j}$

$a_k = 0 \quad |k| > 3$

EXAMPLE #2:-



→ symmetric & even function.

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

SOL:- Periodic $= T$ Fundamental Frequency $= \omega_0 \Rightarrow 2\pi/T$

$K = 0 \quad . \quad a = 0 \qquad t = 0 \qquad -T/2 \leq t < T/2$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\, dt = \frac{1}{T}\left| t \right|_{-T_1}^{T_1}$$

$$= \frac{1}{T}\left[T_1 + T_1\right] \Rightarrow \frac{2T_1}{T} \text{ ans}$$

a_0 is average value of $x(t)$ according to interpretation.

Day/Date

When $k \neq 0$ $\quad x(t)=1$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt$$

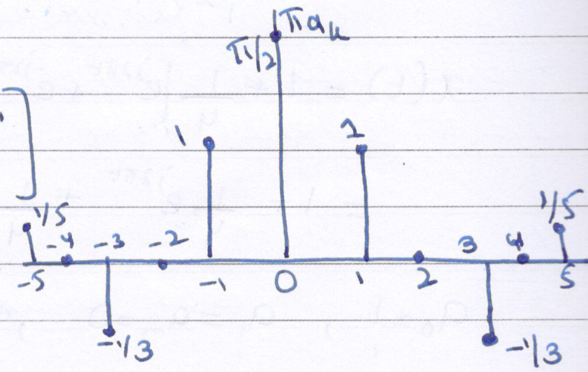
$$= \frac{1}{T}\left[-\frac{e^{-jk\omega_0 t}}{jk\omega_0}\right]_{-T_1}^{T_1} = -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1}$$

$$= \frac{-1}{jk\omega_0 T}\left[e^{-jk\omega_0 T_1} - e^{+jk\omega_0 T_1}\right]$$

we can write it as

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{-e^{-jk\omega_0 T_1} + e^{jk\omega_0 T_1}}{2j}\right]$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \quad \underline{\text{ans.}}$$



Now as $\omega_0 T = 2\pi$

$$a_k = \frac{\cancel{2}\sin(k\omega_0 T_1)}{k\cancel{2}\pi} \Rightarrow \frac{\sin(k\omega_0 T_1)}{k\pi} \quad k \neq 0$$

→ For $T = 4T_1$ $x(t)$ is a square wave That is unity for half the period and zero for half the period.

→ In this case $\omega_0 T_1 = \pi/2$

$$a_k = \frac{\sin(k\pi/2)}{k\pi} \quad k \neq 0$$

while $a_0 = \frac{2T_1}{T} = \frac{2\left(\frac{\pi/2}{\omega_0}\right)}{\frac{2\pi}{\omega_0}}$

$$a_0 = \frac{1}{2} \qquad a_0 = \left(\cancel{2}\,\frac{\pi}{2\omega_0}\cdot\frac{\omega_0}{2\pi}\right) \Rightarrow \frac{1}{2}.$$

Date

FOURIER SERIES :-

→ decomposition of signals as a linear combination of complex exponentials.

→ • the system response easy to be computed
• many general signals out of the input.

Eigen function :-

→ $\phi_k(t) = e^{j\omega_k t}$

$e^{j\omega_k t} \Rightarrow H(\omega_k) e^{j\omega_k t}$ the response is exactly the same form just multiplied by the complex factor.

Proof :- $e^{j\omega_k t} \rightarrow \int_{-\infty}^{\infty} h(\tau) e^{j\omega_k (t-\tau)} d\tau$

$\downarrow$

$e^{j\omega_k t} e^{-j\omega_k \tau}$

$$e^{j\omega_k t} \rightarrow \underset{\text{eigen function}}{e^{j\omega_k t}} \Rightarrow \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_k \tau} d\tau}_{H(\omega_k) \rightarrow \text{eigen value.}}$$

same frequency multiplied by complex factor.

→ This is eigen function property.

→ Because of the Eigen function property, complex exponentials are convenient building blocks.

→ Periodic signals ⇒ Fourier Series
→ Aperiodic signals ⇒ Fourier transform.

C.T. FOURIER SERIES:-

→ Fourier series is a representation of periodic C.T signals.

$x(t) = x(t + T_0)$ → Period.

$\omega_0 = \frac{2\pi}{T_0} \Rightarrow 2\pi f_0$ fundamental frequency.

Day/Date

→ let's ~~deal~~ start with complex exponentials & recognize that there is a complex exponential that has exactly the same period and fundamental frequency as a more general periodic signals.

$$e^{j\omega_0 t} \qquad \therefore T_0 = \frac{2\pi}{\omega_0}$$

→ harmonicall related complex exponentials.

$$e^{jk\omega_0 t} \qquad \therefore \frac{T_0}{k} = \frac{2\pi}{k\omega_0}$$

- k varies then these are harmonically related.

→ Fourier series says that if there is general periodic signal then it can be represented as a linear combination of harmonically related complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow \text{fourier series}$$

→ How we determine fourier series coefficient a_k?

→ How broadic class of signals can be represented this way?

→ refers to as complex exponential form:-

$$a_k = A_k e^{j\theta_k} \rightarrow \text{polar from}$$
$$= B_k + jC_k \rightarrow \text{rectangle form}$$

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j \sin k\omega_0 t$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$$= a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]$$

Trignometric form.

→ only positive frequencies

→ but in complex exponential we have both positive and negative frequencies. So we will use this often.

Day/Date

→ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

→ how we determine coefficients a_k?

$$\int_{T_0} e^{jm\omega_0 t}\, dt = \begin{cases} T_0 & m=0 \\ 0 & m \neq 0 \end{cases}$$ (random one period).

$$= \int_{T_0} \cos m\omega_0 t\, dt + j\int_{T_0} \sin m\omega_0 t\, dt$$

→ if $m \neq 0$ both above signals will be 0.

→ if $m = 0$ the integral of cos will be T_0 and sin will be 0.

→ $\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt = \int_{T_0} e^{-jn\omega_0 t} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, dt$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{T_0} e^{-j(k-n)\omega_0 t}\, dt$$

$$\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_{T_0} e^{j(k-n)\omega_0 t}\, dt$$

$$\left.\begin{matrix} T_0 & \text{if } k=n \\ 0 & \text{if } k\neq n \end{matrix}\right\}$$

$$\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt = a_n T_0$$

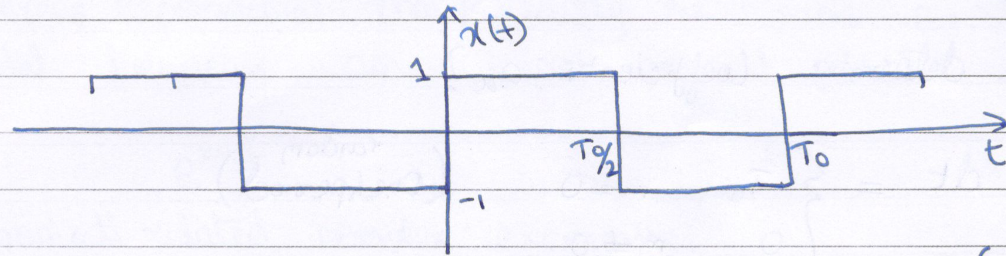
$$\frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt = a_n$$ (Fourier series coefficient a_n or a_n)

Synthesis equation:- $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Analysis equation:- $a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt.$

Day/Date

Example:- ANTISYMMETRIC PERIODIC SQUARE WAVE:-



→ Antisymmetric is referring to odd time function. (about the origin)

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{0} (-1)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T_0}\int_{0}^{T_0/2} (+1)\, e^{-jk\omega_0 t}\, dt$$

after solving integration we are left with

$$a_k = \frac{1}{j\pi k}\left\{1-(-1)^k\right\} \qquad k \neq 0$$

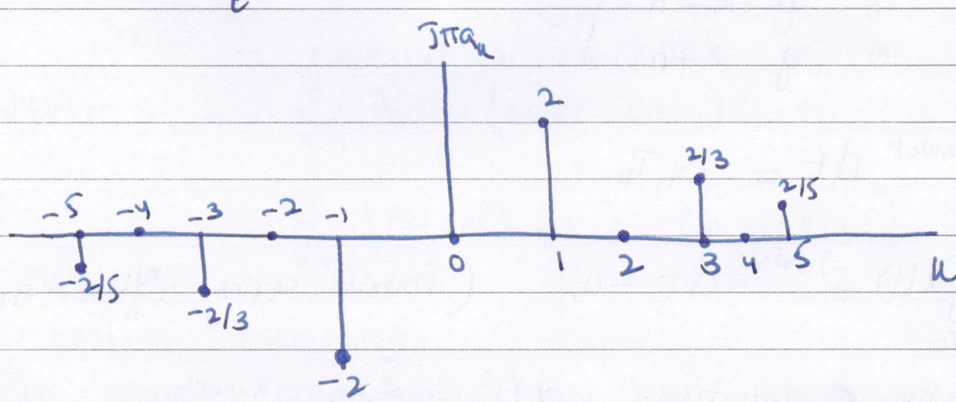
for $k=0$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, \cancelto{1}{e^{-jk\omega_0 t}}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\, dt.$$ average value (it's 0).

$a_0 = 0 \qquad k=0$

$$a_k = \frac{1}{j\pi k}\left\{1-(-1)^k\right\} \qquad k \neq 0$$



Fourier series coefficient

- purely imaginary & • odd.

Date

→ $x(t) = a_0 + \sum_{k=1}^{\infty} 2j a_k \sin k\omega_0 t$ (involves only sine terms).

a_k is imaginary.

Fourier series ~~are~~ for antisymmetric is sine series

→ $x_N(t)$ partial sum.

→ Does $e_N(t)$ decrease as N increases?

CONVERGENCE OF FOURIER SERIES:-

→ If $x(t)$ square integrable i-e integrable is finite.
so the energy goes to 0 as $N \to \infty$.

→ Dirichlet conditions :-

- If $x(t)$ is absolutely integrable then $e_N(t) \to 0$ as $N \to \infty$ except at discontinuties.
-

→ For periodic signal that has no discontinuities, the Fourier series representation converges and equals the original signal at every value of t.

→ For a periodic signal with finite number of discontinuities in each period the Fourier series representation equals the signal everywhere except at the isolated points of discontinuity at which the series converges to the average value of the signal on either side of the discontinuity.