Signal & Systems

Fourier Series-I

9th November 16

Historical Perspective

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History

- In 1822, the French mathematician J.B.J. Fourier had first studied the periodic function and published his famous theorem.
- It states that any periodic signal can be resolved into sinusoidal components.
- Fourier series is the resulting summation of harmonic sinusoid.
- The signal can be in time domain or in frequency domain.
- T can be represented either in the form of infinite trigonometric series or in the form of exponential series.

The Response of LTI Systems

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Introduction

- Based on superposition property of LTI systems, response to any input including linear combination of basic signal is the same linear combination of the individual responses to each of the basic signals.
- Continuous-time and Discrete-time periodic signals are described by Fourier Series.
- Aperiodic signals are described by Fourier Transform.

Response of LTI Systems to Complex Exponentials

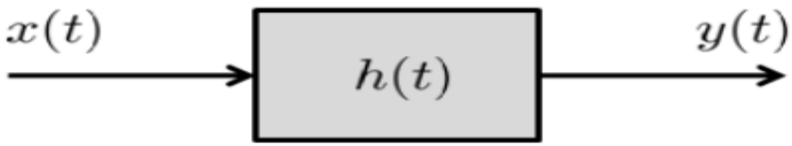
- For analyzing LTI systems, the signals can be represented as a linear combination of basic signals.
- Basic signals possess the following two properties:
  - The set of basic signals can be used to construct a broad and useful class of signals.
  - Should have simple structure in LTI system response.
- Both of these properties are provided by the set of complex exponential signals in continuous and discrete time.
- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.
- For Continuous time: $e^{st} \rightarrow H(s)e^{st}$ where H(s) is a function of s.
- For Discrete time: $z^{n} \rightarrow H(z)z^{n}$ where H(z) is a function of z.

Eigenfunctions of an LTI System

- If the output is a scaled version of its input, then the input function is called an Eigenfunction of the system.
- The scaling factor is called the eigenvalue of the system.

Continuous-Time

❖ Consider an LTI system with impulse response h(t) and input signal x(t).



❖ Suppose that x(t) = e^{st} for some s belongs to C, then the output is given by:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

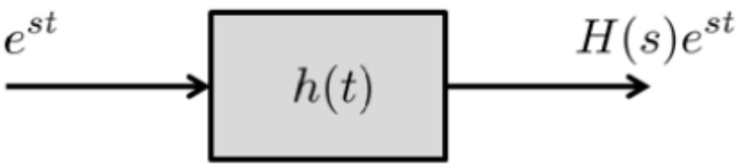
$$= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau$$

$$= e^{st}\left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau\right] = H(s)e^{st} = H(s)x(t)$$

Continuous-Time (cont.)

- Where H(s) is defined as: $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau$

- From the above derivation we see that if the input is $x(t) = e^{st}$, then the output is a scaled version $y(t) = H(s)\, e^{st}$.



- Therefore, using the definition of Eigenfunction, we show that:
  - e^{st} is an Eigenfunction of any continuous-time LTI system
  - H(s) is the corresponding eigenvalue.

Continuous-Time (cont.)

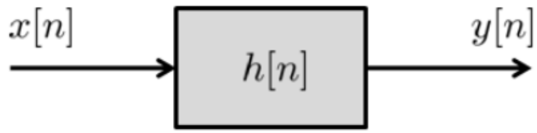
- Considering the subclass of periodic complex exponentials of the $e^{j\omega t}$, ω belongs to R by setting s=jω, then:

$$H(s)\Big|_{s=j\omega} = H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$

- H(jω) is called the frequency response of the system.

Discrete-Time Case

- In parallel manner we can show that complex exponential sequences are Eigenfunctions of discrete-time LTI systems.



- Suppose tat the impulse response is given by h[n] and the input is $x[n]=z^n$, then the output y[n] is:

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

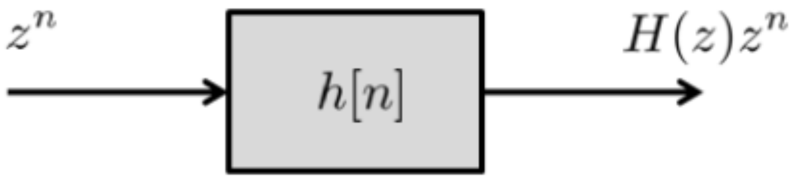
$$=\sum_{k=-\infty}^{\infty}h[k]z^{[n-k]}=z^n\sum_{k=-\infty}^{\infty}h[k]z^{-k}=H(z)z^n$$

- Where:

$$H(z)=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$$

Discrete-Time Case (cont.)

- This result indicates:
  - z^n is an Eigenfunction of a discrete-time LTI system
  - $H(z)$ is the corresponding eigenvalue.



- Considering the subclass of periodic complex exponentials $e^{-j(2\pi/N)n}$ by setting $z= e^{j2\pi/N}$, we have:

$$H(z)\big|_{z=e^{j\Omega}} = H\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

$$where \quad \Omega = \frac{2\pi}{N}$$

- And $H(e^{j\Omega})$ is called the frequency response of the system.

Importance of Eigenfunction

- The usefulness of Eigenfunctions can be seen from an example.
- Lets consider a signal x(t):

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

- According to the Eigenfunction analysis , the output of each complex exponential is:

$$e^{s_1 t} \rightarrow H(s_1) e^{s_1 t}$$

$$e^{s_2 t} \rightarrow H(s_2) e^{s_2 t}$$

$$e^{s_3 t} \rightarrow H(s_3) e^{s_3 t}$$

- From the superposition property the response to the sum is the sum of the responses, so that:

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

Importance of Eigenfunction (cont.)

- The result implies that if the input is a linear combination of complex exponentials, the output of an LTI system is an infinite sum of complex exponentials.
- More generally, if x(t) is an infinite sum of complex exponentials,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$$

- Then the output is:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$$

- Similarly for discrete-time signals, if:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n$$

then

$$y[n] = \sum_{k=-\infty}^{\infty} a_k H(z_k) z_k^n$$

Importance of Eigenfunction (cont.)

- This is an important observation, because as long as we can express a signal x(t) as a linear combination of Eigenfunctions, then the output y(t) can be easily determined by looking at the transfer function. Same goes for discrete-time.
- The transfer function is fixed for an LTI system.

Fourier Series of Continuous-Time Periodic Signals

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Fourier Series of Continuous-Time

- According to the definition of periodic signals: x(t) = x(t+T) with fundamental period T and fundamental frequency $\omega_0=2\pi/T$.
- We have also discussed two basic signals, the sinusoidal signal: $x(t)=\cos\omega_0 t$ and the periodic complex exponential $x(t) = e^{j\omega o t}$.
- Both of these signals are periodic with fundamental frequency ω_0 and the fundamental period $T=2\pi/\omega_0$.
- Harmonically related complex exponentials:

$$\phi_k\left(t\right)=e^{jk\omega_0 t}=e^{jk(2\pi/T)t},k=0,\pm1,\pm2,......$$

- Each harmonic has fundamental frequency which is multiple of ω_0.
- A Linear combination of harmonically related complex exponentials:

$$x\left(t\right)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}a_k e^{jk(2\pi/T)t}$$

Fourier Series of Continuous-Time (cont.)

- Above equation is also periodic with period T.
- k=±1 have fundamental frequency ω_0 (first harmonic)
- k=±N have fundamental frequency $N\omega_0$ (Nth harmonic)

Continuous-Time Fourier Series Coefficients

- Theorem: The continuous-time Fourier series coefficients a_k of the signal:

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \textit{Synthesis} \quad \textit{Equation}$$

- Is given by:

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt, \quad \textit{Analysis} \quad \textit{Equation}$$

- Proof:
- Let us consider the signal:

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- If we multiply $e^{-jn\omega_0 t}$ on both sides, then we have:

$$x(t)e^{-jn\omega_0 t}=\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right]e^{-jn\omega_0 t}=\sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

Continuous-Time Fourier Series Coefficients (cont.)

- Integrating both sides from 0 to T yields: (T is the fundamental period of x(t))

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \left[\sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\right] dt$$

$$= \sum_{k=-\infty}^{\infty}\left[a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt\right]$$

- Use Euler's formula:

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos\left(\left(k-n\right)\omega_0 t\right)dt + j\int_0^T \sin\left(\left(k-n\right)\omega_0 t\right)dt$$

- For k≠n, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period (T/|k-n|)

Continuous-Time Fourier Series Coefficients (cont)

❖ Therefore:

$$\frac{1}{T}\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} 1 & if \quad k=n \\ 0 & otherwise \end{cases}$$

❖ This result is known as the orthogonality of complex exponentials.

❖ Using above equation we have:

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = Ta_n$$

❖ Which is equivalent to:

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

❖ Dc or constant component of x(t):

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

Example #1

- Consider the signal: $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$
- The period of x(t) is T=2, so the fundamental frequency is $\omega_0 = 2\pi/T = \pi$.
- Recall Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, we have:

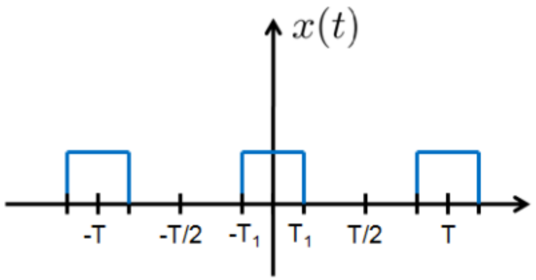
$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$$a_0 = 1, \quad a_1 = a_{-1} = 0, \quad a_2 = a_{-2} = \frac{1}{4}, \quad a_3 = \frac{1}{2j}, \quad a_{-3} = -\frac{1}{2j}$$

$$and \quad a_k = 0 \quad otherwise$$

Example #2

❖ Periodic Rectangular Wave:



❖ Let us determine the Fourier series coefficients of the following signal:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \dfrac{T}{2} \end{cases}$$

❖ The Fourier series coefficients when k≠0 are:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt$$

Example #2 (cont.)

$$= \frac{-1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1}$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin\left(k\omega_0 T_1\right)}{k\omega_0 T}$$

$$if \quad k = 0, \quad then$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

Convergence of the Fourier Series

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Existence of Fourier Series

- To understand the validity of Fourier Series representation, lets examine the problem of approximation a given periodic signal x(t) by a linear combination of a finite number of harmonically related complex exponentials.
- That is by finite series of the form:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- Let $e_N(t)$ denote the approximation error; i.e.,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- The criterion that we will use is the energy in the error over one period:

$$E_N(t) = \int_T \left| e_N(t) \right|^2 dt$$

Existence of Fourier Series (cont.)

- To achieve min E_N, one should define:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

- As N increases, E_N decreases and as $N \rightarrow \infty$ E_N is zero.
- If $a_k \rightarrow \infty$ the approximation will diverge.
- Even for bounded a_k the approximation may not be applicable for all periodic signals.

Convergence Conditions of Fourier Series Approximation

- Energy of signal should be a finite in a period:

$$\int_T |x(t)|^2 \, dt < \infty$$

  - This condition only guarantees $E_N \rightarrow 0$.
  - It does not guarantee that x(t) equals to its Fourier series at each moment t.
- Dirichlet Conditions:
  - Over any period x(t) must be absolutely integrable.
  - In any finite interval of time x(t) is of bounded variation, i.e., there are no more than a finite number of maxima and minima during any single period of the signal.
  - In any finite interval of time, there are only a finite number of discontinuities.

Thankyou

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