Lecture Notes
11th November 2016

Day/Date FRIDAY / 11th NOV, 16 LECTURE # 8

Why do we use Fourier Series?

→ F.S is just a means to represent a periodic signal as an infinite sum of sine wave components.

→ The main reason to use F.S is that we ~~that~~ can better analyse a signal in another domain rather in the original ~~form~~ domain.

D.T Fourier Series:-

→ When k is changed by any integer multiple of N, the identical sequence is generated.

DETERMINATION OF DISCRETE-TIME FOURIER SERIES COEFFICIENTS

→ We have a sequence $x[n]$ that is periodic with fundamental period N.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

→ Multiply both sides by $e^{-j\omega_0 mn}$:-

$$e^{-j\omega_0 mn} x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} e^{-j\omega_0 mn}$$

→ Take summation on $n = \langle N\rangle$:

$$\sum_{n=\langle N\rangle} e^{-j\omega_0 mn} x[n] = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-m)\omega_0 n}$$

$$\sum_{n=\langle N\rangle} e^{-j\omega_0 mn} x[n] = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-m)\omega_0 n}$$

→ Use the sum of geometrical series rule i.e:

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \end{cases}$$

→ $\therefore \sum_{n=\langle N\rangle} e^{-j\omega_0 mn} x[n] = \sum_{k=\langle N\rangle} a_k N \delta[k-m] \Rightarrow N a_m$

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→ $a_m = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jw_0 mn}$

→ $a_k = a_{k+N}$

EXAMPLE #1:-

$$x[n] = \sin w_0 n$$

SOL:-

→ This is the discrete-time counterpart of the signal
$$x(t) = \sin w_0 t.$$

→ For the case when $2\pi / w_0$ is an integer N, that is when
$$w_0 = \frac{2\pi}{N}$$

→ $x[n]$ is periodic with fundamental period N. and we obtain a result that is exactly analogous to the continuous time case.

→ Expanding the signal as a sum of two complex exponentials, we get.
$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$

→ we see by inspection
$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}$$

→ remaining coefficients over the interval of summation are zero.

→ As described earlier, these coefficients repeat with period N, thus $a_{N+1} = \frac{1}{2j}$ and $a_{N-1} = -\frac{1}{2j}$.

→ For example:- F.S coefficients for this example with $N=5$ are shown below.

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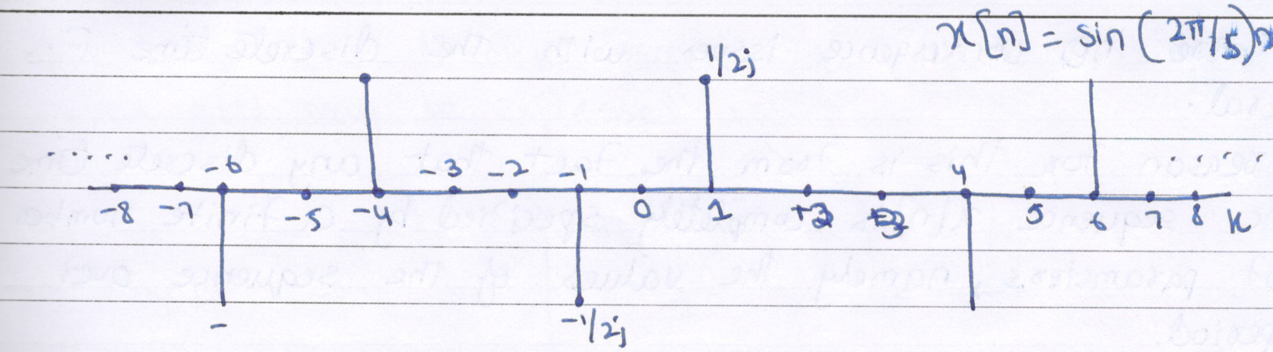

→ They repeat periodically.

EXAMPLE #2:-

SOL:-

→ $x[n] = 1$ for $-N_1 \leq n \leq N_1$, we choose the length-N interval of summation to include the range $-N_1 \leq n \leq N_1$. The coefficie are given:-

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

→ let $m = n + N_1$, then above equation becomes:-

$$a_k = \frac{1}{N} \sum_{n=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

$$a_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \left[\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right]$$

$$= \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \ldots\ldots$$

→ For $k = 0, \pm N, \pm 2N, \ldots\ldots$, we have

$$a_k = \frac{2N_1 + 1}{N}.$$

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→ There are no convergence issues with the discrete time F.S in general.

→ The reason for this is from the fact that any discrete time periodic sequence $x[n]$ is completely specified by a finite number N of parameters, namely the values of the sequence over one period.

→ There is no Gibbs phenomenon for DT. Fourier series.

→ The sum of DT FS is finite.

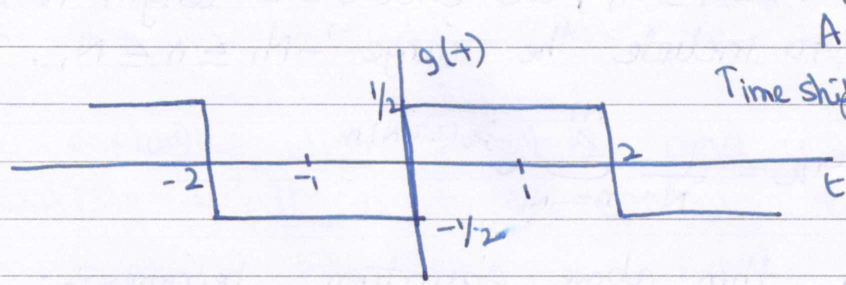
→ No approximate is made so no convergence issue.

EXAMPLE # 3:-

Linearity:-

$Ax_1(t) + Bx_2(t) \Leftrightarrow Aa_k + Bb_k$

Time shift:- $x(t-t_0) \Leftrightarrow a_k e^{-jk\omega_0 t_0}$



SOL:-

$T = 4$ & $T_1 = 1$

$$g(t) = x(t-1) - \frac{1}{2} \Rightarrow (1)$$

→ According to time shift property, if Fourier Series coefficients of $x(t)$ are denoted by a_k, the F.S coefficients of $x(t-1)$ may be expressed as:-

$$b_k = a_k e^{-jk\pi/2} \qquad \therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} \Rightarrow \pi/2$$

→ The Fourier series coefficients of the dc offset in $g(t)$ i-e the term $-1/2$ on the right hand side of equ (1) are given by:-

$$c_k = \begin{cases} 0 & \text{for } k \neq 0 \\ -1/2 & \text{for } k = 0. \end{cases}$$

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→ Applying the linearity property we conclude that the coefficient for $g(t)$ may be expressed as:-

$$d_k = \begin{cases} a_k e^{-jk\pi/2} & \text{for } k \neq 0 \\ a_0 - \frac{1}{2} & \text{for } k = 0 \end{cases}$$

where each a_k may now be replaced by ~~the~~ :-

$$a_k = \frac{\sin(\pi k/2)}{k\pi} e^{jk\pi/2} \text{, then we have}$$

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2} & \text{for } k \neq 0 \\ 0 & \text{for } k = 0 \end{cases}$$

EXAMPLE #4:-

SOL:-

→ Finding the Fourier Series coefficients a_k of the sequence $x[n]$.

→ $N = 5$

→ The signal $x[n]$ may be viewed as the sum of the square wave $x_1[n]$ with Fourier series coefficients b_k and $x_2[n]$ with Fourier series coefficients c_k.

$$a_k = b_k + c_k$$

→ The fourier series coefficients of $x_1[n]$ is:

$$b_k = \begin{cases} \frac{1}{5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)} & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5} & \text{for } k = 0, \pm 5, \pm 10 \ldots \end{cases}$$

Day/Date $\frac{3}{5}+1 \Rightarrow \frac{3+5}{5} \Rightarrow 8/5$

→ The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient.

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1$$

→ Since the discrete time fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5.

→ The remaining coefficients of $x_2[n]$ must be zero, because $x_2[n]$ contains only a dc component.

→ then,

$$a_k = \begin{cases} b_k = \frac{1}{5}\frac{\sin(k3\pi k/5)}{\sin(\pi k/5)} & \text{for } k \neq 0, \pm 5, \pm 10 \cdots \\ \frac{8}{5}, & \text{for } k = 0, \pm 5, \pm 10 \cdots \end{cases}$$

$$a_k = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, k \neq 0, \pm 5, \pm 10 \cdots$$