Signal & Systems

Fourier Series-II

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Convergence of the Fourier Series

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Existence of Fourier Series

- To understand the validity of Fourier Series representation, lets examine the problem of approximation a given periodic signal x(t) by a linear combination of a finite number of harmonically related complex exponentials.
- That is by finite series of the form:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- Let $e_N(t)$ denote the approximation error; i.e.,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- The criterion that we will use is the energy in the error over one period:

$$E_N(t) = \int_T \left| e_N(t) \right|^2 dt$$

Existence of Fourier Series (cont.)

- To achieve min E_N, one should define:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

- As N increases, E_N decreases and as $N \rightarrow \infty$ E_N is zero.
- If $a_k \rightarrow \infty$ the approximation will diverge.
- Even for bounded a_k the approximation may not be applicable for all periodic signals.

Convergence Conditions of Fourier Series Approximation

- Energy of signal should be a finite in a period:

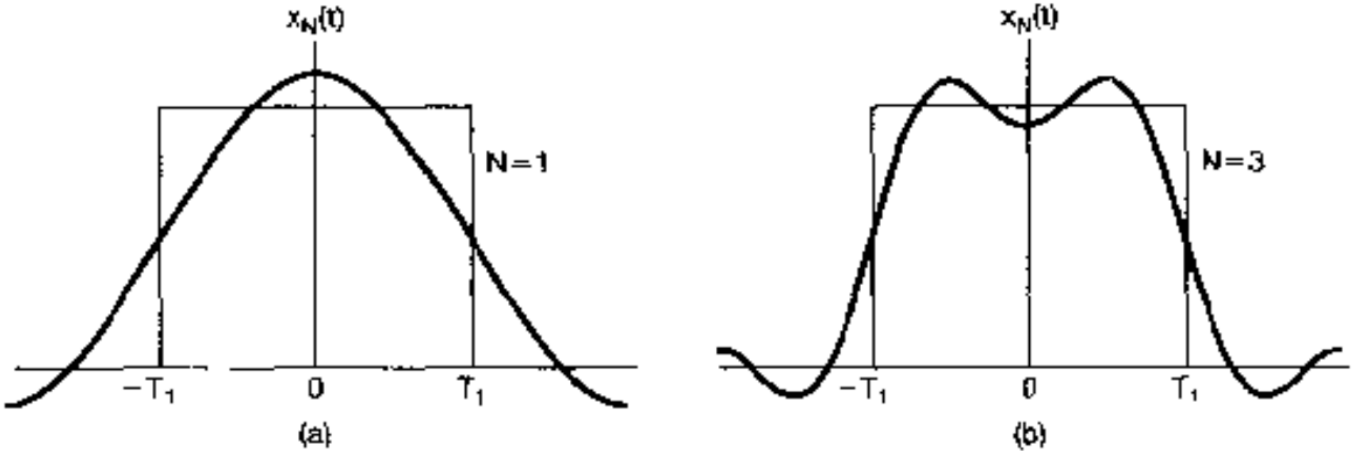
$$\int_T |x(t)|^2 \, dt < \infty$$

  - This condition only guarantees $E_N \rightarrow 0$.
  - It does not guarantee that x(t) equals to its Fourier series at each moment t.

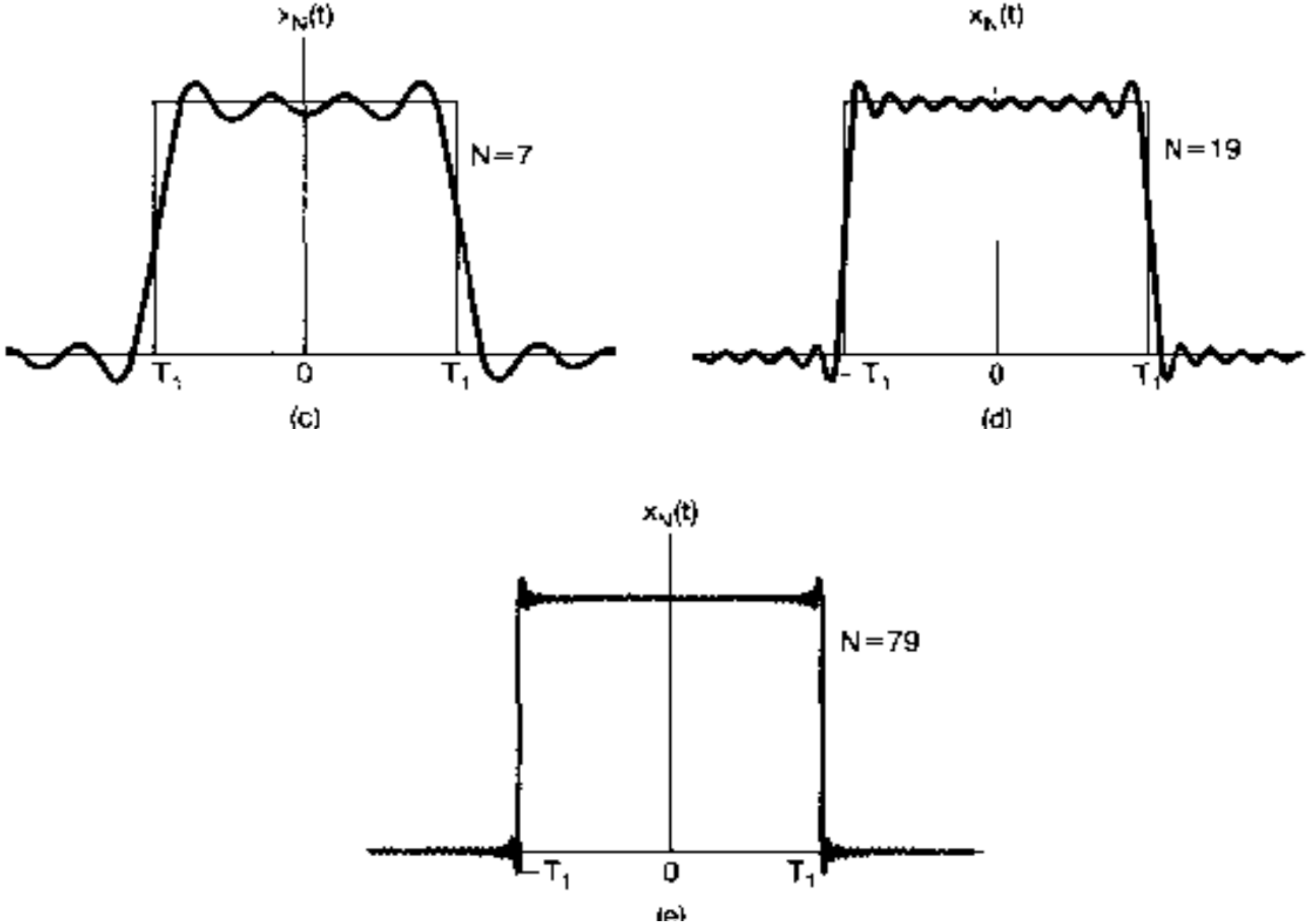
- Dirichlet Conditions:
  - Over any period x(t) must be absolutely integrable.
  - In any finite interval of time x(t) is of bounded variation, i.e., there are no more than a finite number of maxima and minima during any single period of the signal.
  - In any finite interval of time, there are only a finite number of discontinuities.

Gibbs Phenomenon

- Near a point where x(t) has a jump discontinuity, the partial sums x_N (t) of a Fourier series exhibit a substantial overshoot near these endpoints.
- An increase in N will not diminish the amplitude of the overshoot, although with increasing N the overshoot occurs over smaller and smaller intervals.
- This phenomenon is known as Gibbs Phenomenon.



Gibbs Phenomenon (cont.)



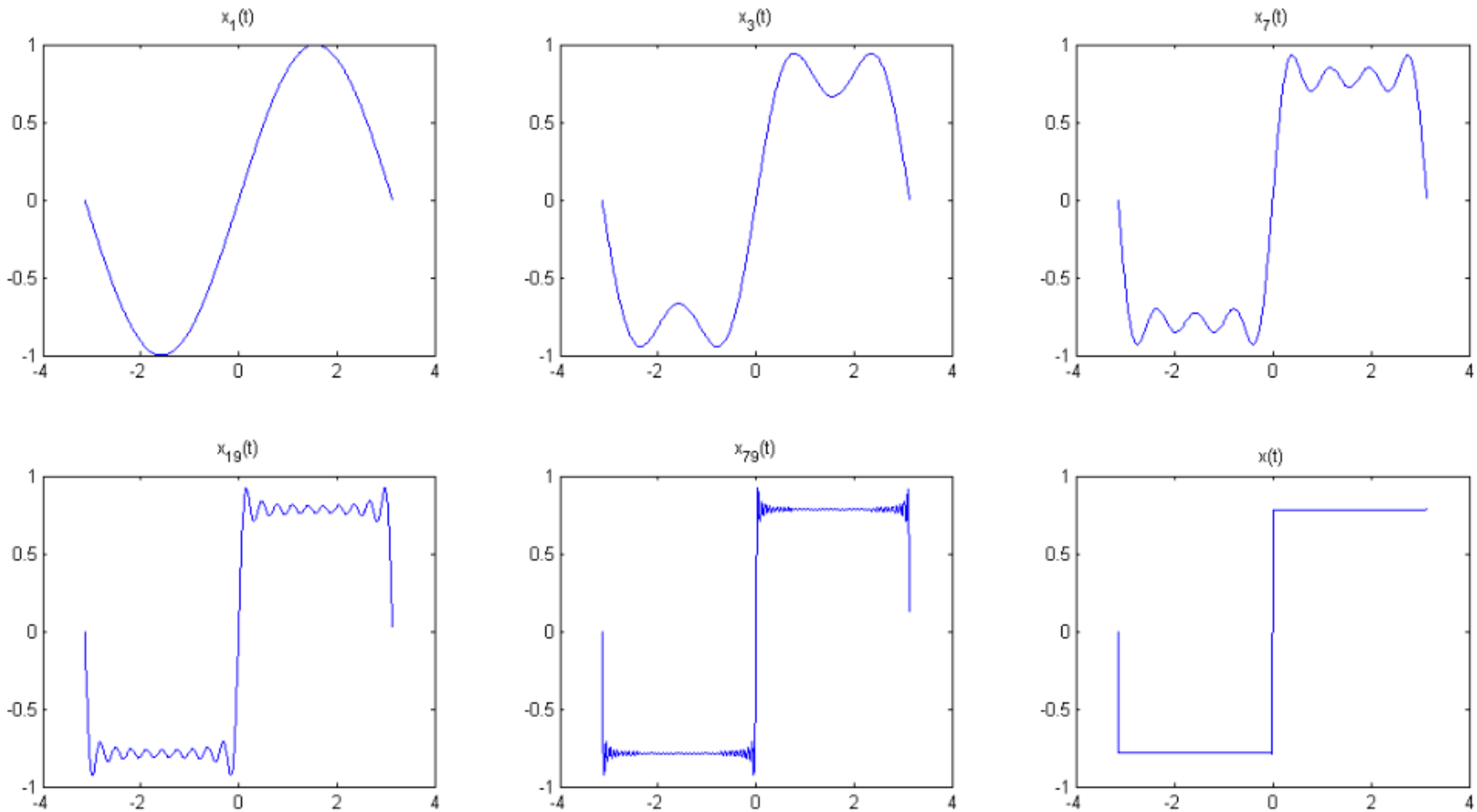
How many Fourier Series Coefficients are sufficient?

- If we define: $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$
- Then $x_N(t)$ is an approximation of x(t). As N approaches to infinity, then $x_N(t)$ approaches to x(t).
- Therefore the number of Fourier series coefficients depends on the accuracy that we want to achieve.
- Typically, the number N is chosen such that the residue of the approximation:

$$\int_{-\infty}^{\infty} \left| x(t) - x_N(t) \right|^2 dt \leq \varepsilon$$

- For some target error level ε.

How many Fourier Series Coefficients are sufficient?



Fourier Series Representation of Discrete-Time Periodic Signals

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Fourier Series Representation of DT

- The Fourier series representation of a discrete-time periodic signal is finite as opposed to the infinite series representation required for continuous-time periodic signals.

Linear Combinations of Harmonically Related Complex Exponentials

- A discrete-time signal x[n] is periodic with period N if: x[n] = x[n+N].
- The fundamental period is the smallest positive N and the fundamental frequency is $\omega_0 = \frac{2\pi}{N}$.
- The set of all discrete-time complex exponential signals that are periodic with period N is given by:

$$\phi_k\left[n\right] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, ...$$

- All of these signals have fundamental frequencies that are multiples of 2π/N and thus are harmonically related.
- There are only N distinct signals in the set this is because the discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. That is: $\phi_k\left[n\right] = \phi_{k+rN}\left[n\right]$

Linear Combinations of Harmonically Related Complex Exponentials (cont.)

- The representation of periodic sequences in terms of linear combinations of the sequences $\Phi_k[n]$ is:

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}$$

- Since the sequences $\Phi_k[n]$ are distinct over a range of N successive values of k, the summation in above equation need include terms over this range.
- Thus the summation is on k as k varies over a range of N successive integers beginning with any value of k.
- We indicate this by expressing the limits of the summation as k=<N>. That is:

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

Discrete-Time Fourier Series Coefficients

- Assuming x[n] is square-summable i.e., $\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty$ or x[n] satisfies the Dirichlet conditions.
- In this case we have:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \qquad \textit{Synthesis} \quad \textit{Equation}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}, \qquad \textit{Analysis} \quad \textit{Equation}$$

- As in continuous time, the discrete-time Fourier series coefficient a_k are often referred to as the spectral coefficients of x[n].
- These coefficients specify a decomposition of x[n] into a sum of N harmonically related complex exponentials.

Example #1

- Consider the signal:

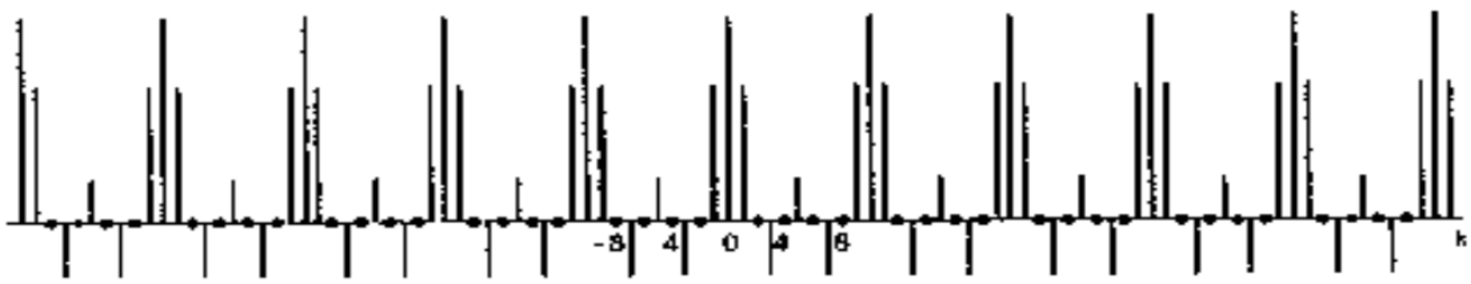
$$x[n] = \sin \omega_0 n$$

- Which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$.
- x[n] is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers.

Example #2

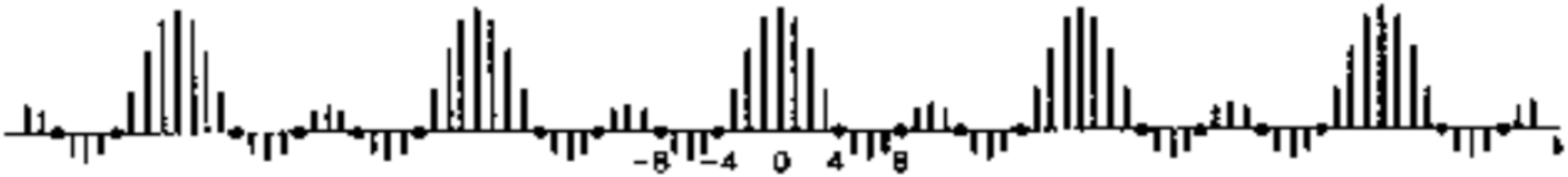
- Lets consider the discrete-time periodic square wave shown below:

- Determine the discrete-time Fourier series coefficient.
- Solution:
- The coefficients a_k for $2N_1+1=5$ are sketched for N=10,20 and 40 in figures below:



(a)

Example #2 (cont.)



(b)

(c)

Properties of Fourier Series Coefficients

Linearity

- For continuous-time Fourier series, we have:

$$x_1(t) \leftrightarrow a_k \quad and \quad x_2(t) \leftrightarrow b_k$$

$$Ax_1(t) + Bx_2(t) \leftrightarrow Aa_k + Bb_k$$

- For Discrete-time case, we have:

$$x_1[n] \leftrightarrow a_k \quad and \quad x_2[n] \leftrightarrow b_k$$

$$Ax_1[n] + Bx_2[n] \leftrightarrow Aa_k + Bb_k$$

Time Shift

$$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$$

$$x[n-n_0] \leftrightarrow a_k e^{-jk\Omega_0 n_0}$$

- Proof: Let us consider the Fourier series coefficient b_k of the signal $y(t)=x(t-t_0)$.

$$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-j\omega_0 t}\,dt$$

- Letting $\tau = t-t_0$ in the integral, we obtain:

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}\,dt = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}\,dt$$

$$where \quad x(t) \leftrightarrow a_k. \quad Therefore,$$

$$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$$

Time Reversal

$$x(-t) \leftrightarrow a_{-k}$$

$$x[-n] \leftrightarrow a_{-k}$$

- Proof: Consider a signal y(t) = x(-t). The Fourier series representation of x(-t) is:

$$x(-t) = \sum_{-\infty}^{\infty} a_k e^{-jk2\pi t/T}$$

- Letting k = -m, we have:

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

- Thus:

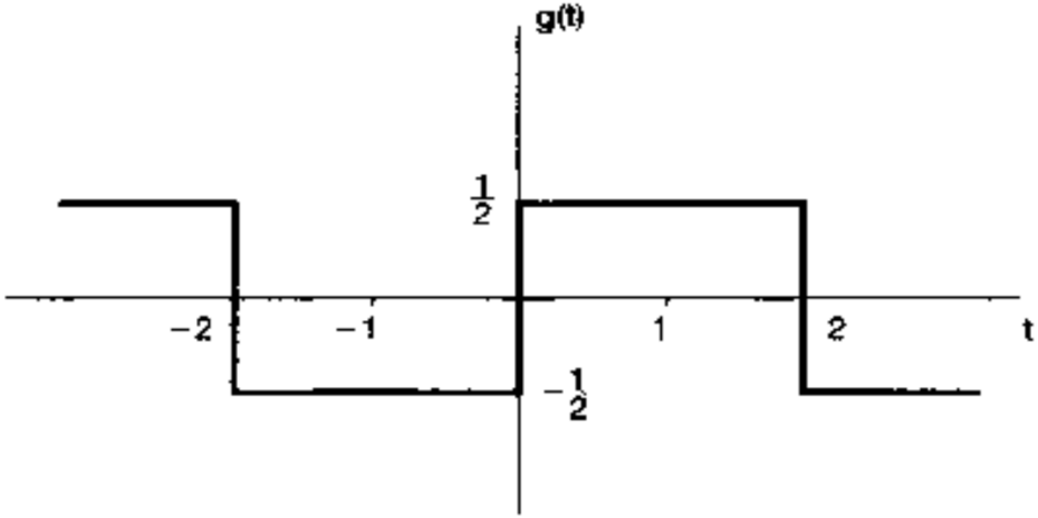
$$x(-t) \leftrightarrow a_{-k}$$

Time Scaling

- Time scaling is an operation that in general changes the period of the underlying signal.
- Specifically if x(t) is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

Example #3

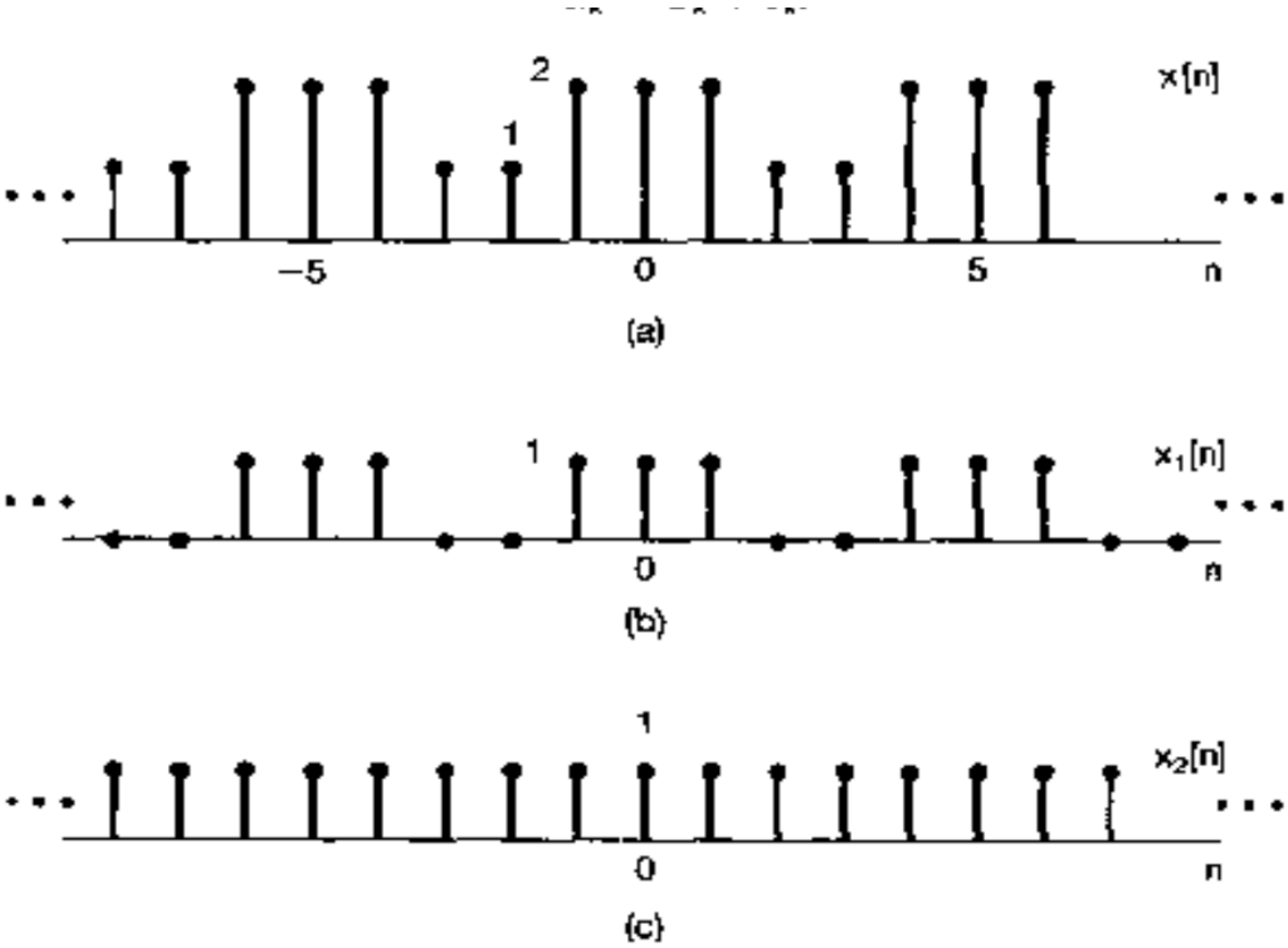
- Consider the signal g(t) with a fundamental period of 4 shown below:



- Apply Time shift property.
- Apply Linearity property.

Example #4

❖ Consider the signal shown below:



Thankyou

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