Lecture Notes
14th November 2016

Day/Date MONDAY / 14th NOV, 16 LECTURE #9

EXAMPLE #1:-

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

$$h(t) = e^{-t} u(t)$$

SOL:-

→ To calculate the Fourier series coefficients of the output y(t) we first compute the frequency response:-

$$H(j\omega) = \int_0^\infty e^{-\tau} e^{-j\omega\tau} \, d\tau = \int_0^\infty e^{-\tau + (-j\omega\tau)} \, d\tau$$

$$= \int_0^\infty e^{-\tau - j\omega\tau} \, d\tau = \int_0^\infty e^{-\tau(1+j\omega)} \, d\tau$$

$$= \frac{1}{1+j\omega} e^{-\tau} \Big|_0^\infty = -\frac{1}{1+j\omega} \left[e^{-\infty} - e^{-0}\right]$$

$$H(j\omega) = -\frac{1}{1+j\omega}\left[-e^0\right] \Rightarrow \frac{1}{1+j\omega}$$

→ The output is:-

$$y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t}$$

→ where $b_k = a_k H(jk\omega_0) = a_k H(jk2\pi)$, so that

$$b_0 = 1, \quad b_1 = \frac{1}{4}\left[\frac{1}{1+j2\pi}\right], \quad b_{-1} = \frac{1}{4}\left[\frac{1}{1-j2\pi}\right]$$

$$b_2 = \frac{1}{2}\left[\frac{1}{1+j4\pi}\right], \quad b_{-2} = \frac{1}{2}\left[\frac{1}{1-j4\pi}\right]$$

$$b_3 = \frac{1}{3}\left[\frac{1}{1+j6\pi}\right], \quad b_{-3} = \frac{1}{3}\left[\frac{1}{1-j6\pi}\right]$$

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EXAMPLE #2:-

$$h[n] = a^n u[n] \quad , -1 < a < 1$$

$$x[n] = \cos\left(\frac{2\pi n}{N}\right)$$

SOL:-

→ Writing the signal $x[n]$ in Fourier series form as:-

$$x[n] = \frac{1}{2} e^{j(2\pi/N)n} + \frac{1}{2} e^{-j(2\pi/N)n}$$

→ The transfer function is:-

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \Rightarrow \frac{1}{1-ae^{-j\omega}}$$

⇒ The Fourier series for the output:-

$$y[n] = \frac{1}{2} H(e^{j2\pi/N}) e^{j(2\pi/N)n} + \frac{1}{2} H(e^{-j2\pi/N}) e^{-j(2\pi/N)n}$$

$$= \frac{1}{2}\left(\frac{1}{1-ae^{-j\omega}}\right) e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-ae^{-j\omega}}\right) e^{-j(2\pi/N)n}$$

EXERCISE PROBLEMS:-

PROBLEM #1:-

Express $x(t)$ in the form:-

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

SOL:-

$x(t)$ is real valued.

$T = 8 \quad , a_1 = a_{-1} = 2, \quad a_3 = a^*_{-3} = 4j$

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→ Using the Fourier series synthesis equ:-

$$x(t) = a_1 e^{j(2\pi/T)t} + a_{-1} e^{-j(2\pi/T)t} + a_3 e^{j3(2\pi/T)t} + a_{-3} e^{-j3(2\pi/T)t}$$

$$= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4j e^{j3(2\pi/8)t} - 4j e^{-j3(2\pi/8)t}$$

$$= 2\left[e^{j(\pi/4)t} + e^{-j(\pi/4)t}\right] + 4j\left[e^{j(6\pi/8)t} - e^{-j(6\pi/8)t}\right]$$

$$= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{6\pi}{8}t\right)$$

$$x(t) \Rightarrow 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)$$

PROBLEM #2:-

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$\omega_0 = ?$

Fourier series coefficients $a_k = ?$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

SOL:-

The given signal is:-

$$x(t) = 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t}$$

$$= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t}$$

→ From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 \Rightarrow \frac{\pi}{3}$.

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→ The non-zero Fourier series coefficients of $x(t)$ are:

$$a_0 = 2, \quad a_2 = a_{-2} = \frac{1}{2}, \quad a_5 = a^*_{-5} = -2j$$

PROBLEM #3:-

$x[n]$ = real and odd periodic signal

$N = 7$, $a_k = ?$

$a_{15} = j$, $a_{16} = 2j$, $a_{17} = 3j$

$a_0 = ?$ $a_{-1} = a_{-2} = a_{-3} = ?$

SOL:-

Since the Fourier series coefficients repeat every N, we have

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$$a_1 = a_{15} \quad , \; a_2 = a_{16} \quad , \; \& \; a_3 = a_{17}$$

→ Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd.
Therefore, $a_0 = 0$ and.

$$a_1 = -a_{-1} \quad , \quad a_2 = -a_{-2} \quad , \quad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j \quad , \; a_{-2} = -2j \; , \; a_{-3} = -3j$$

PROBLEM #4:-

$x[n] =$ real & even signal , $N = 10$, a_k

$a_{11} = 5$

$$\frac{1}{10} \sum_{n=0}^{9} |x[n]|^2 = 50$$

$$x[n] = A \cos(Bn + C) = ?$$

SOL:-

→ Since the Fourier series coefficients repeat every $N = 10$, we have $a_1 = a_{11} = 5$.

→ Furthermore, since $x[n]$ is real and even, a_k is also real and even.

→ Therefore, $a_1 = a_{-1} = 5$.

→ We are also given that $\frac{1}{10} \sum_{n=0}^{9} |x[n]|^2 = 50$

→ Using Parseval's relation,

$$\sum_{k=\langle N \rangle} |a_k|^2 = 50$$

$$\sum_{k=-1}^{8} |a_k|^2 = 50$$

$$|a_{-1}|^2 + |a_1|^2 + |a_0|^2 + \sum_{k=2}^{8} |a_k|^2 = 50$$

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$$|5|^2 + |5|^2 + a_0^2 + \sum_{k=2}^{8} |a_k|^2 = 50$$

$$a_0^2 + \sum_{k=2}^{8} |a_k|^2 = 0$$

→ Therefore, $a_k = 0$ for $k = 2, \ldots\ldots, 8$.

→ Now using the synthesis equation, we have.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-1}^{8} a_k e^{j\frac{2\pi}{10}kn}$$

$$= 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n}$$

$$x[n] \Rightarrow 10\cos\left(\frac{\pi}{5}n\right)$$

PROBLEM #5:-

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \frac{\sin(4\omega)}{\omega}$$

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

$T = 8$, $y(t) = ?$

SOL:-

→ let us first evaluate the Fourier series coefficients of $x(t)$.

→ Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd.

→ Therefore, $a_0 = 0$. Now,

$$a_k = \frac{1}{8}\int_0^8 x(t)e^{-j(2\pi/8)kt}\,dt$$

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$$= \frac{1}{8}\int_0^4 e^{-j(2\pi/8)kt}\,dt - \frac{1}{8}\int_4^8 e^{-j(2\pi/8)kt}\,dt$$

$$a_k = \frac{1}{j\pi k}\left[1 - e^{-j\pi k}\right]$$

→ clearly the above expression evaluates to zero for all even values of k.

→ Therefore,

$$a_k = \begin{cases} 0 & k = 0, \pm 2, \pm 4, \ldots \\ \frac{2}{j\pi k} & k = \pm 1, \pm 3, \pm 5, \ldots \end{cases}$$

→ When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\cancel{2}\pi}{\cancel{8}4} \Rightarrow \frac{\pi}{4}$.

→ since a_k is nonzero only for odd values of k, we need to evaluate the above summation only for odd k.

→ Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

→ is alway zero for odd values of k. There fore,

$$y(t) = 0$$

PROBLEM #8:-

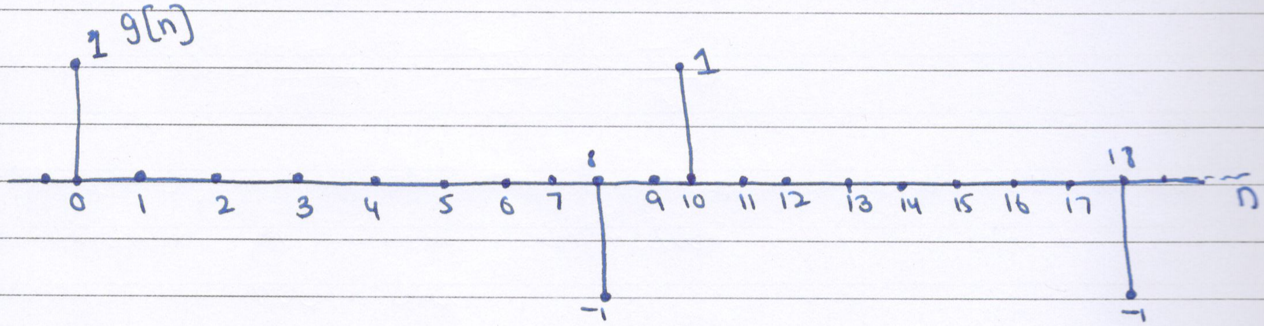
$$x[n] = \begin{cases} 1 & , 0 \le n \le 7 \\ 0 & , 8 \le n \le 9 \end{cases} \quad , N=10$$

$$g[n] = x[n] - x[n-1]$$

SOL:-

a) Show that $g[n]$ has a fundamental period of 10.

$g[n]$ is as shown below.



Clearly, $g[n]$ has a fundamental period of 10.

b) Fourier series coefficients of $g[n]$ = ?

$$b_k = \frac{1}{10}\left[1 - e^{-j(2\pi/10)8k}\right]$$

c) a_k for $k \neq 0$ = ?

Since $g[n] = x[n] - x[n-1]$, the FS coefficients a_k & b_k must be related as:

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$$x[n] - x[n-1] \xleftrightarrow{FS} a_k\left(1 - e^{-jk(2\pi/N)}\right)$$

$$b_k = a_k - e^{-j(2\pi/10)k}\, a_k$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-j(2\pi/10)k}}$$

$$= \frac{(1/10)\left[1 - e^{-j(2\pi/10)8k}\right]}{1 - e^{-j(2\pi/10)k}}$$