Signal & Systems

Fourier Series-III

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Properties of Fourier Series

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Properties of Continuous-Time Fourier Series

- **Multiplication:**
  - Suppose that x(t) and y(t) are both periodic with period T and that:

$$x(t) \leftrightarrow a_k$$

$$v(t) \leftrightarrow b_k$$

  - Since the product x(t) y(t) is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for x(t) and y(t). The result is:

$$x(t)y(t) \leftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

  - The sum on the R.H.S may be interpreted as the Discrete-time convolution of the sequence x(t) and y(t).

Properties of Continuous-Time Fourier Series (cont.)

- **Conjugation and Conjugate Symmetry:**
  - Real $x(t) \leftrightarrow a_{-k} = a_k^*$ (conjugate symmetric)
  - Real & Even $x(t) \leftrightarrow a_k = a_k^*$ (real and even a_k)
  - Real & Odd $x(t) \leftrightarrow a_k = -a_k^*$ (purely imaginary and odd a_k), $a_0=0$
  - Even part of $x(t) \leftrightarrow \mathrm{Re}\{a_k\}$
  - Odd part of $x(t) \leftrightarrow j\,\mathrm{Im}\{a_k\}$

Properties of Continuous-Time Fourier Series (cont.)

- **Parseval's Relation:**
  - Parseval's relation for continuous-time periodic signal is:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

  - Where a_k are the Fourier series coefficients of x(t) and T is the period of the signal.
  - L.H.S of the above equation is the average power (i.e., energy per unit time) in one period of the periodic signal x(t).
  - Also:

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

  - So that $|a_k|^2$ is the average power in the kth harmonic component of x(t).
  - Thus Parseval's relation states that the total average power in a periodic signals equals the sum of the average powers in all of its harmonic components.

Properties of Discrete-Time Fourier Series

- **Multiplication:**
  - In discrete-time, suppose that:

$$x[n] \leftrightarrow a_k$$
$$and$$
$$y[n] \leftrightarrow b_k$$

  - Are both periodic with period N. then the product x[n] y[n] is also periodic with period N.
  - Its Fourier coefficients d_k are given by:

$$x[n]y[n] \leftrightarrow d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

  - The result is a periodic convolution between the FS sequences.
  - w[n] is periodic with N.

Properties of Discrete-Time Fourier Series (cont.)

- **First Difference:**
  - If x[n] is periodic with period N, then so is y[n], since shifting x[n] or linearly combining x[n] with another periodic signal whose period is N always results in a periodic signal with period N.
  - Also, if:

$$x[n] \leftrightarrow a_k$$

  - Then the Fourier coefficients corresponding to the first difference of x[n] may be expressed as:

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-jk(2\pi/N)}\right) a_k$$

Properties of Discrete-Time Fourier Series (cont.)

- **Parseval's Relation:**
  - Parseval's relation for discrete-time periodic signals is given by:

$$\frac{1}{N}\sum_{n=\langle N\rangle}\left|x[n]\right|^2 = \sum_{n=\langle N\rangle}\left|a_k\right|^2$$

  - The average power in a periodic signal = the sum of the average power in all of its harmonic components.

Properties of Continuous-Time Fourier Series

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Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t} a_k$
Frequency shifting	$e^{jM\omega_0 t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t)$, $\alpha > 0$ (Periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau) y(t-\tau) d\tau$	$Ta_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$

Properties of Continuous-Time Fourier Series (cont.)

Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \text{Re}\{a_k\} = \text{Re}\{a_{-k}\} \\ \text{Im}\{a_k\} = -\text{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x(t)$ real and even $x(t)$ real and odd $\begin{cases} x_e(t) = Ev\{x(t)\} & [x(t)\ real] \\ x_e(t) = Od\{x(t)\} & [x(t)\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\text{Re}\{a_k\}$ $j\,\text{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$	

Properties of Discrete-Time Fourier Series

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamenta l frequency $\omega_0 = 2\pi$	$\begin{matrix} a_k \\ b_k \end{matrix}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$e^{-jk(2\pi/N)t} a_k$
Frequency shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}

Properties of Discrete-Time Fourier Series (cont.)

Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & if\ n\ is\ a\ multiple\ of\ n \\ 0, & if\ n\ is\ a\ multiple\ of\ n \end{cases}$ (Periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} viewed\ as\ periodic \\ with\ period\ mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=[N]} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
Differentiation	$x[n]-x[n-1]$	$\left(1-e^{-jk(2\pi/N)}\right)a_k$
Integration	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{1-e^{-jk(2\pi/N)}}\right)a_k$

Properties of Discrete-Time Fourier Series (cont.)

Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	$x[n]$ real and even $x[n]$ real and odd $\begin{cases} x_e[n] = Ev\{x[n]\} \quad [x[n]\ real] \\ x_e[n] = Od\{x[n]\} \quad [x[n]\ real] \end{cases}$	a_k real and even a_k purely imaginary and odd $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{T}\sum_{n=<N>} \lvert x[n] \rvert^2 = \sum_{n=<N>} \lvert a_k \rvert^2$	

Fourier Series & LTI Systems

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Fourier Series & LTI Systems

- The response of a continuous-time LTI system with impulse response h(t) to a complex exponential signal e^{st} is the same complex exponential multiplied by a complex gain:

$$y(t) = H(s)e^{st}$$

$$\text{where}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$$

- In particular, for s=jω, the output is $y(t)=H(j\omega)e^{j\omega t}$.
- The complex functions H(s) and H(jω) are called the system function (or transfer function) and the frequency response, respectively.

Fourier Series & LTI Systems (cont.)

- By superposition, the output of an LTI system to a periodic signal represented by a Fourier series:

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad is \quad given \quad by$$

$$y(t)=\sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

- That is, the Fourier series coefficients b_k of the periodic output y(t) are given by: $b_k = a_k H(jk\omega_0)$
- Similarly, for discrete time signals and systems, response h[n] to a complex exponential signal $e^{j\omega n}$ is the same complex exponential multiplied by a complex gain: $y[n]=H(jk\omega_0)e^{jk\omega_0 n}$

$$where$$

$$H(e^{j\omega})=\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Example #1

- Suppose that the periodic signal $x(t)=\sum_{k=-3}^{3} a_k e^{jk2\pi t}$ with $a_0=1$, $a_1=a_{-1}=1/4$, $a_2=a_{-2}=1/2$, and $a_3=a_{-3}=1/3$ is the input signal to an LTI system with impulse response $h(t)=e^{-t}u(t)$.

Example #2

- Consider an LTI system with impulse response $h[n]=\alpha^n u[n]$, $-1<\alpha<1$, and with the input:

$$x[n] = \cos\left(\frac{2\pi n}{N}\right)$$

Exercise Problems

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Problem #1

- A continuous-time periodic signal x(t) is real valued and has a fundamental period T=8. The non-zero Fourier series coefficients for x(t) are:

$$a_1 = a_{-1} = 2, a_3 = a_{-3}^* = 4j$$

- Express x(t) in the form:

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

Problem #2

- For the continuous-time periodic signal:

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

- Determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Problem #3

- Let x[n] be a real and odd periodic signal with period N=7 and Fourier coefficients a_k. Given that:

$$a_{15} = j, a_{16} = 2j, a_{17} = 3j$$

- Determine the values of a_0, a_{-1}, a_{-2} and a_{-3} .

Problem #4

- Suppose we are given the following information about a signal x[n]:
  - 1. x[n] is a real and even signal.
  - 2. x[n] has period N=10 and Fourier coefficients a_k .
  - 3. $a_{11}=5$
  - 4. $\frac{1}{10}\sum_{n=0}^{9}\left|x[n]\right|^2 = 50$
  - Show that x[n] = Acos (Bn+C) , and specify numerical values for the constants A, B and C.

Problem #5

- Consider a continuous-time LTI system whose frequency response is:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt = \frac{\sin(4\omega)}{\omega}$$

- If the input to this system is a periodic signal:

$$x(t) = \begin{cases} 1 & 0 \le t < 4 \\ -1 & 4 \le t < 8 \end{cases}$$

- With period T=8, determine the corresponding system output y(t).

Problem #6

- Let:

$$x[n]=\begin{cases} 1, & 0 \le n \le 7 \\ 0, & 8 \le n \le 9 \end{cases}$$

- Be a periodic signal with fundamental period N=10 and Fourier series coefficients a_k. Also let:

$$g[n]=x[n]-x[n-1]$$

- (a): Show that g[n] has a fundamental period of 10.
- (b): Determine the Fourier series coefficient of g[n].
- (c): Using the Fourier series coefficients of g[n] and the first-difference property, determine a_k for $k \neq 0$.

Thankyou

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