Lecture Notes
21st November 2016

Day/Date MONDAY/21st, NOV 16 LECTURE - 10

FOURIER TRANSFORM:-

EXAMPLE #1:-

$$x[n] = a^n u[n] \quad , |a| < 1$$

SOL:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

Using Geometric series formula:- $S = \sum_{k=0}^{\infty} r^k \Rightarrow \frac{1}{1-r}$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

⇒ Graph is shown in slides. All of these functions are periodic in ω with period 2π.

EXAMPLE #2:-

$$x[n] = a^{|n|} \quad , |a| < 1$$

SOL:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$m = -(-\infty)$

Substitute $m = -n$ in first summation

$$X(e^{j\omega}) = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{m=1}^{\infty} (a e^{j\omega})^m + \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

Using geometric series, we have.

Day/Date

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{(1-ae^{j\omega}) + ae^{j\omega}(1-ae^{-j\omega})}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

$$= \frac{1-\cancel{ae^{j\omega}} + \cancel{ae^{j\omega}} - a^2 e^{j\omega-j\omega}}{1-ae^{j\omega}-ae^{-j\omega}+a^2e^{-j\omega+j\omega}} \quad \therefore e^{j\omega-j\omega} = e^0 \Rightarrow 1$$

$$X(e^{j\omega}) = \frac{1-a^2}{1+a^2-2a\cos\omega}$$

Example # 3:-

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad \omega_0 = \frac{2\pi}{5}$$

Sol:-

Using equation of periodicity: $X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$

$$X(e^{j\omega}) = \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\omega-\frac{2\pi}{5}-2\pi l\right) + \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\omega+\frac{2\pi}{5}-2\pi l\right)$$

$$= \frac{1}{\cancel{2}}\times\cancel{2}\sum_{l=-\infty}^{\infty}\pi\delta\left(\omega-\frac{2\pi}{5}-2\pi l\right) + \frac{1}{\cancel{2}}\times\cancel{2}\sum_{l=-\infty}^{\infty}\pi\delta\left(\omega+\frac{2\pi}{5}-2\pi l\right)$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty}\pi\delta\left(\omega-\frac{2\pi}{5}-2\pi l\right) + \sum_{l=-\infty}^{\infty}\pi\delta\left(\omega+\frac{2\pi}{5}-2\pi l\right)$$

That is:-

$$X(e^{j\omega}) = \pi\delta\left(\omega-\frac{2\pi}{5}\right) + \pi\delta\left(\omega+\frac{2\pi}{5}\right), \quad -\pi \le \omega < \pi$$

Day/Date

Example #4:-

$$h[n] = \delta[n-n_0].$$

Sol:-

The frequency response is:-

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n-n_0) e^{-j\omega n} \Rightarrow e^{-j\omega n_0}$$

→ Thus for any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is:

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Example #5:-

$$x[n] = \delta[n] + \delta[n-1] + \delta[n+1].$$

Sol:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\delta[n] + \delta[n-1] + \delta[n+1]\right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1] e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{j\omega} \Rightarrow 1 + 2\cos\omega$$

Example #6:-

$$x[n] = x_1[n]\, x_2[n]$$

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}, \quad x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$

Sol:-

From multiplication property : $Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$
we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ where the integral can be taken over any interval of length 2π.

Day/Date

→ Choosing the interval $-\pi < \theta < \pi$.

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

→ Above equation resembles aperiodic convolution, except for the fact that the ~~interval~~ integration is limited to $-\pi < \theta < \pi$.

→ However, we can convert the equation into an ordinary convolution by defining:

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \text{for } -\pi < \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

→ Then replacing $X_1(e^{j\theta})$ by $\hat{X}_1(e^{j\theta})$ and using the fact that $\hat{X}_1(e^{j\theta})$ is zero for $|\theta| > \pi$, we have

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$