Signal & Systems

DTFT-I

21st November 16

Discrete Time Fourier Transform

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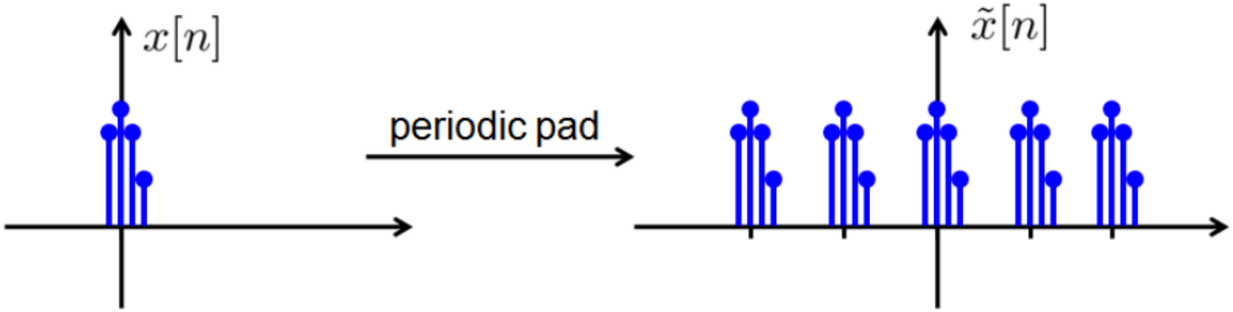
Development of the Discrete-Time Fourier Transform

- In deriving discrete-time Fourier Transform we have three key steps:
- Step#1:
  - Consider an aperiodic discrete-time signal x[n]. We pad x[n] to construct a periodic signal x'[n].



- Step#2:
  - Since x'[n] is periodic, by discrete-time Fourier series we have:

$$x'[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

  - Where a_k is:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x'[n] e^{-jk(2\pi/N)n}$$

Development of the Discrete-Time Fourier Transform (cont.)

- Here, $\omega_0 = 2\pi/N$.
- Now note that x'[n] is a periodic signal with period N and the non-zero entries of x'[n] in a period are the same as the non-zero entries of x[n].
- Therefore, it holds that:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x'[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$$

- If we define:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Then:

$$a_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}X\left(e^{jk\omega_0}\right)$$

Development of the Discrete-Time Fourier Transform (cont.)

- Step#3:
  - Putting above equation in discrete-time Fourier series equation, we have:

$$x'[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$= \sum_{k=\langle N\rangle} \left[\frac{1}{N} X\left(e^{jk\omega_0}\right) \right] e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N\rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0, \quad \omega_0 = \frac{2\pi}{N}$$

  - As $N \rightarrow \infty, \omega_0 \rightarrow 0$, so the area becomes infinitesimal small and sum becomes integration and x'[n]=x[n],so above equation becomes,

$$x'[n] = \frac{1}{2\pi} \sum_{k=\langle N\rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0 \rightarrow \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

Development of the Discrete-Time Fourier Transform (cont.)

- Hence, the Discrete time Fourier transform pair:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- The first equation is referred to as synthesis equation and second one as analysis equation.
- $X(e^{j\omega})$ is referred to as the spectrum of x[n].

Is $X(e^{j\omega})$ Periodic?

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Why is $X(e^{j\omega})$ Periodic?

- The continuous time Fourier transform $X(j\omega)$ is aperiodic in general but the discrete time Fourier transform $X(e^{j\omega})$ is always periodic.
- To prove this, let us consider the discrete-time Fourier transform, here we want to check whether:

$$X\left(e^{j\omega}\right) = X\left(e^{j(\omega+2\pi)}\right)?$$

$$X\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\left(e^{-j2\pi}\right)^{n} = X\left(e^{j\omega}\right)$$

- Because $(e^{-j2\pi})^n = 1^n = 1$, for any integer n. Therefore, $X(e^{j\omega})$ is periodic with period 2π.

Why is $X(e^{j\omega})$ Periodic? (cont.)

- Now, let us consider the continuous-time Fourier transform, and check the periodicity for it,

$$X(j\omega) = X(j(\omega + 2\pi))?$$

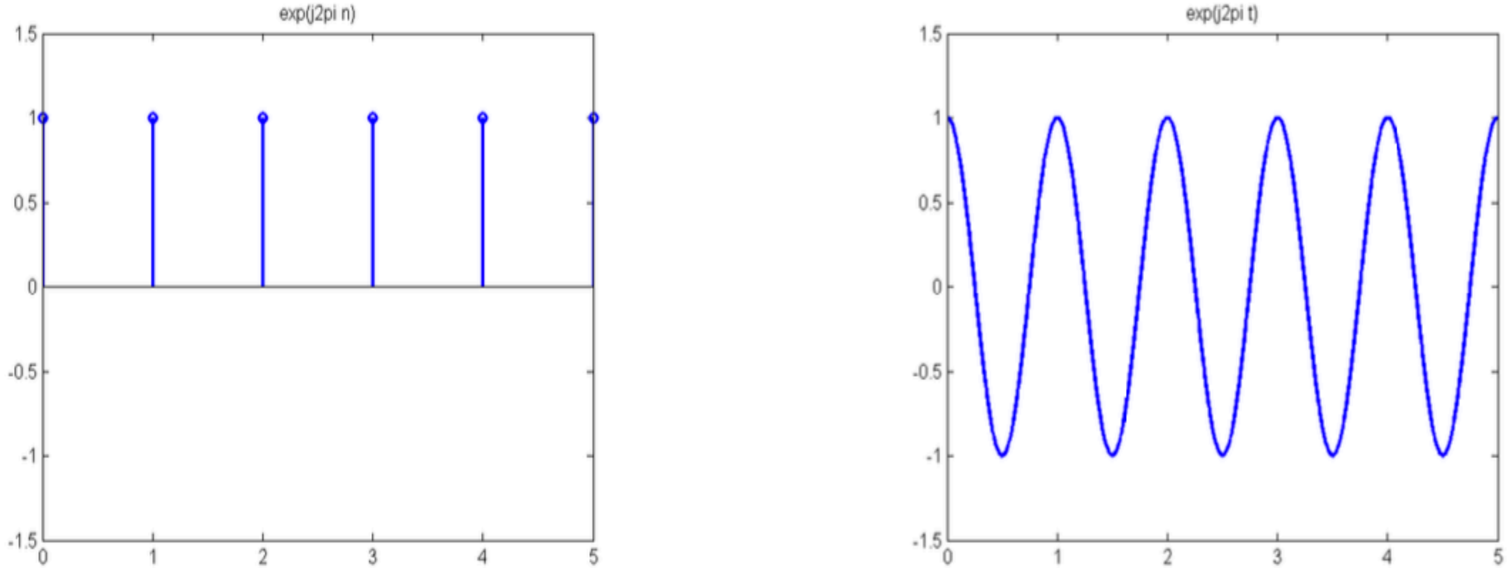
$$X(j(\omega + 2\pi)) = \int_{-\infty}^{\infty} x(t) e^{-j(\omega+2\pi)t}\, dt = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} (e^{-j2\pi})^t\, dt$$

- Here t is a real number and is from -∞ to ∞. But $e^{-j2\pi t} \neq 1$ unless t is an integer and in the case of discrete time n is always an integer.
- Therefore:

$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} (e^{-j2\pi})^t\, dt \neq \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$X(j(\omega + 2\pi)) \neq X(j\omega)$$

Why is $X(e^{j\omega})$ Periodic? (cont.)



(a) $(e^{j2\pi})^n = 1$ for all n, because n is integer. (b) $(e^{j2\pi})^t \neq 1$ unless t is an integer.

Example #1

- Consider the signal:

$$x[n] = a^n u[n], \quad |a| < 1$$

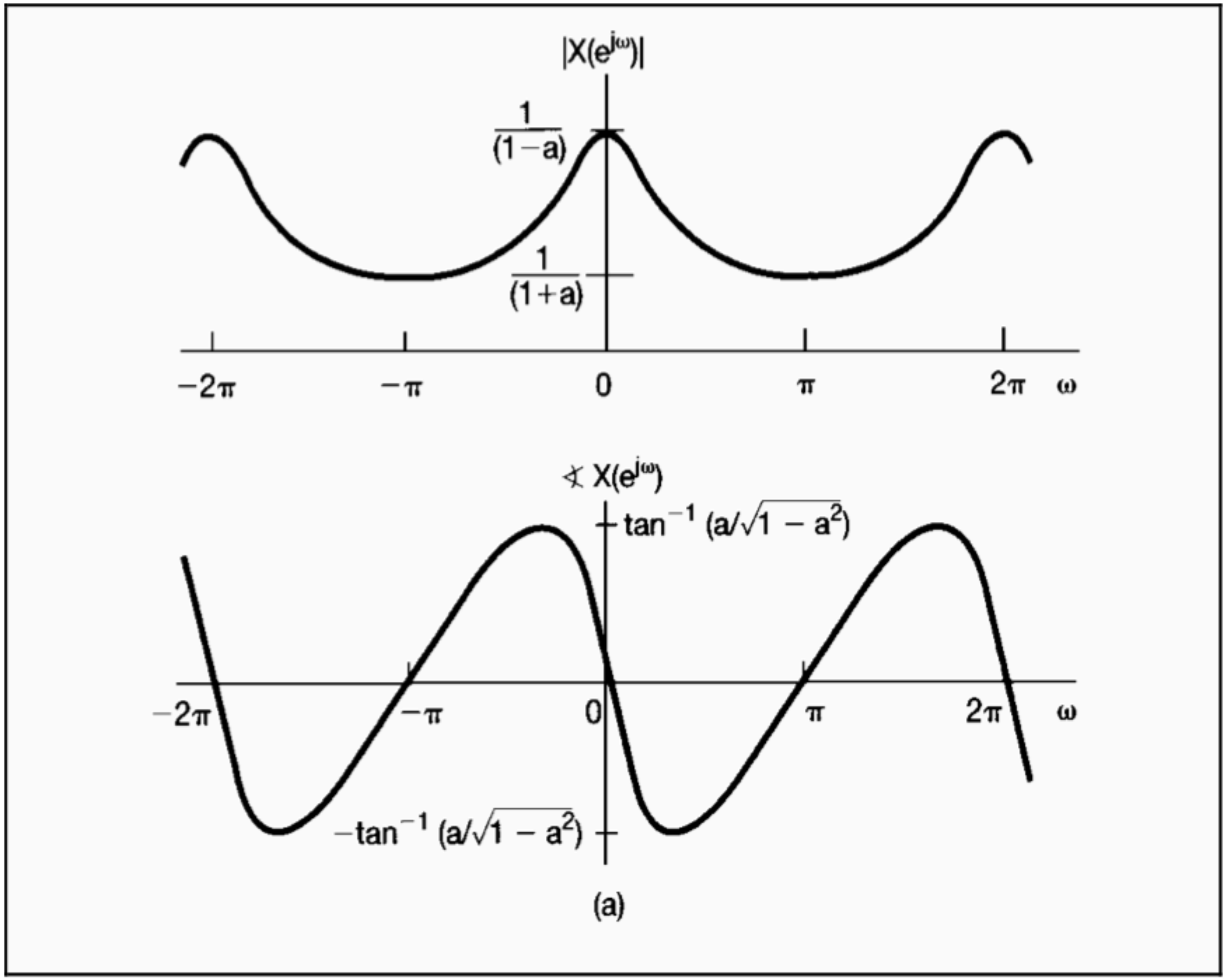
- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n}$$

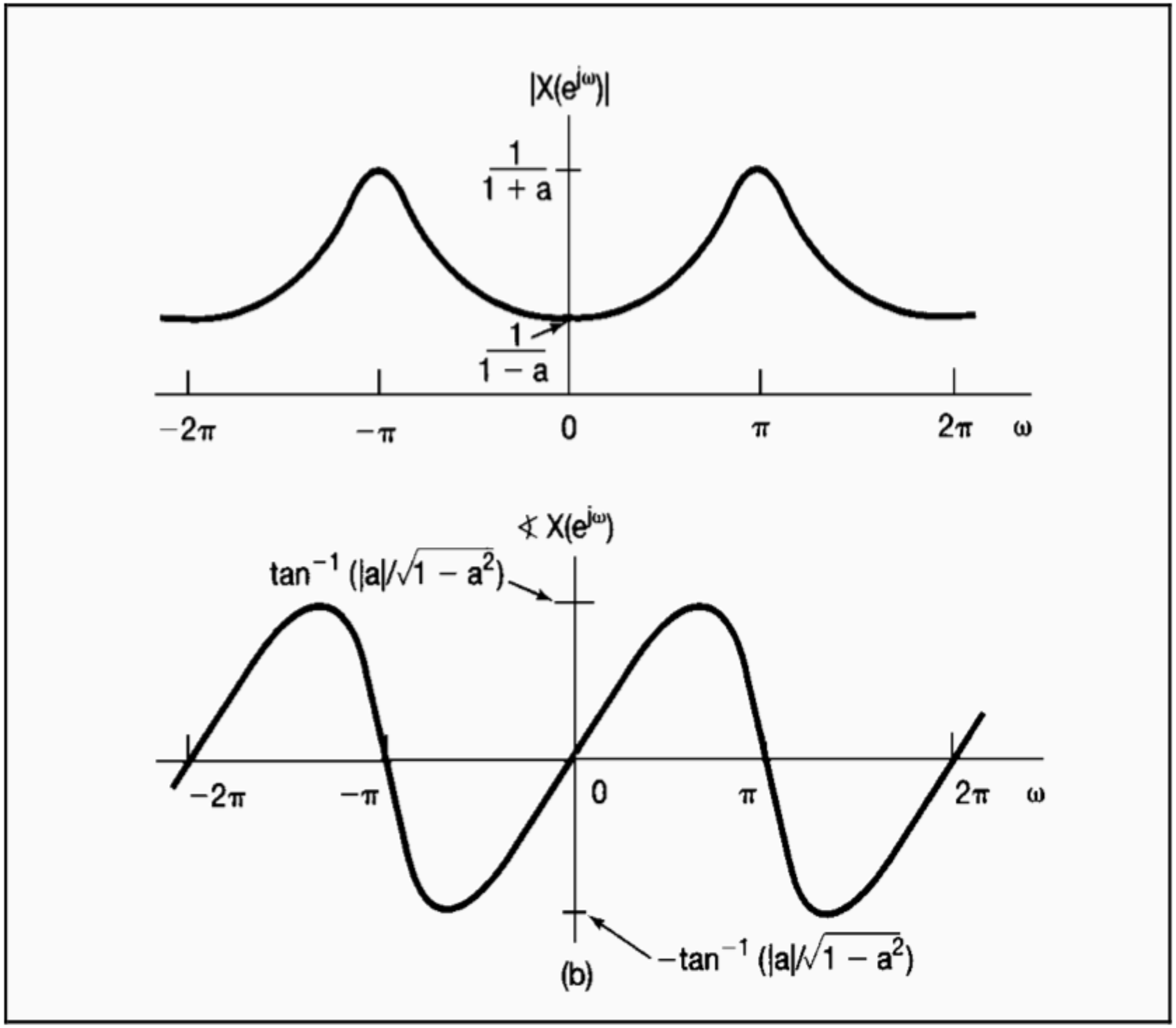
$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

- The magnitude and phase for this example are shown in the figure below, where a>0 and a<0 are shown in figure a and b.

Example #1 (cont.)



Example #1 (cont.)



Example #2

- Consider the signal:

$$x[n] = a^{|n|}, \quad |a| < 1$$

- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

- Let m=-n in the first summation we obtain,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{m=1}^{\infty} a^{m} e^{j\omega m}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{n} + \sum_{m=1}^{\infty} \left(ae^{j\omega}\right)^{m}$$

Example #2 (cont.)

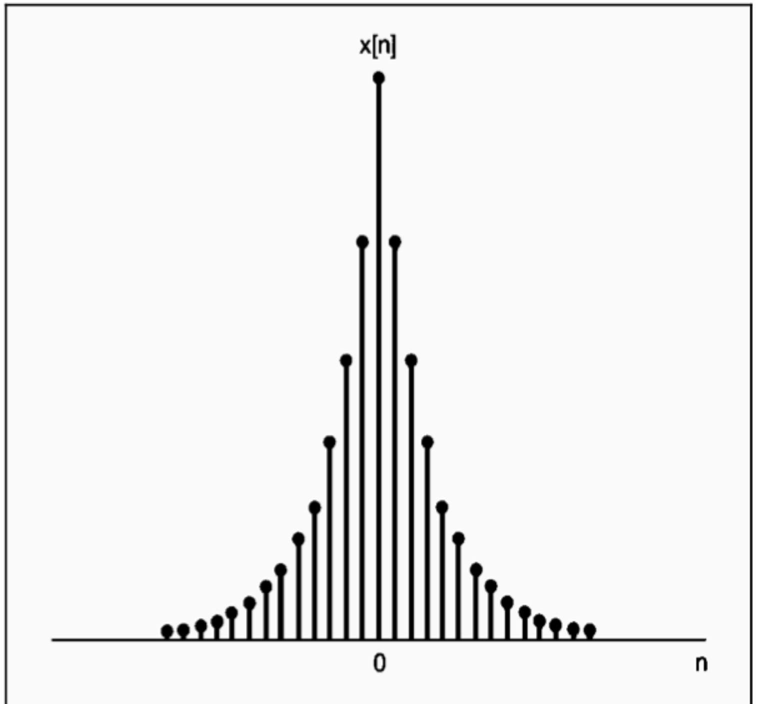
❖ Both of these summations are infinite geometric series that we can evaluate in closed form, yielding:

$$X\left(e^{j\omega}\right) = \frac{1}{1-ae^{j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

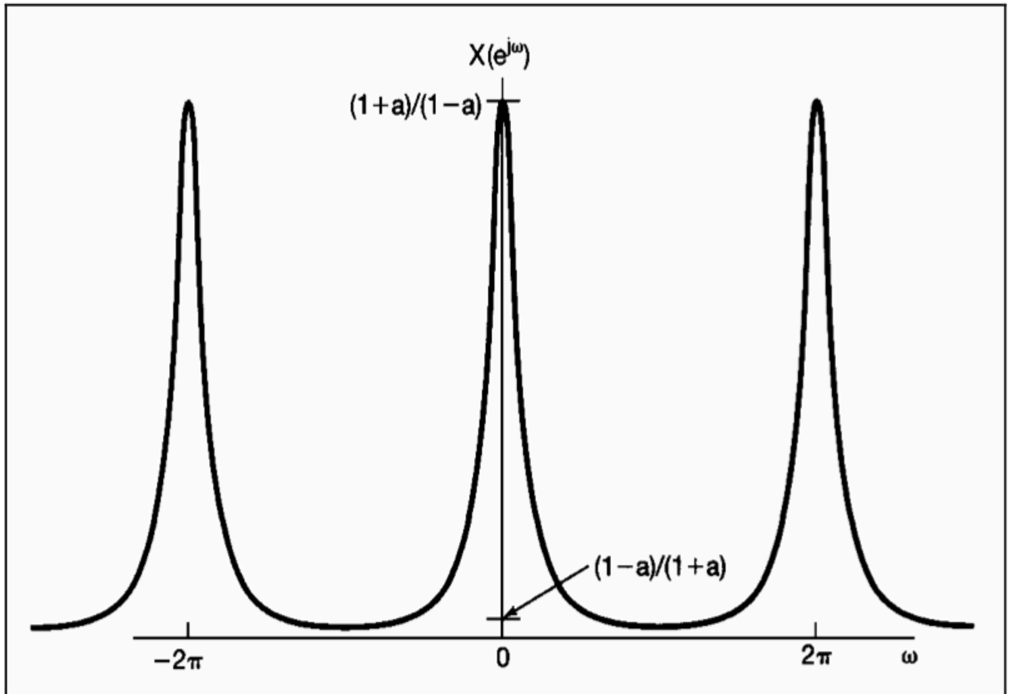
$$= \frac{1-a^2}{1-2a\cos\omega + a^2}$$

❖ Figure (a) below is signal x[n] = $a^{|n|}$ and figure (b) is its Fourier transform (0 < a < 1)

Example #2 (cont.)



(a)



(b)

The Fourier Transform for Periodic Signals

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Periodic Signals

- For a periodic discrete-time signal:

$$x[n] = e^{j\omega_0 n}$$

- The discrete-time Fourier transform must be periodic in ω with period 2π.
- Then the Fourier transform of x[n] should have impulses at ω_0 , $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on.
- In fact, the Fourier transform of x[n] is the impulse train:

$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Now consider a periodic sequence x[n] with period N and with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

Periodic Signals (cont.)

- In this case, the Fourier transform is:

$$X\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$$

- So that the Fourier transform of a periodic signal can directly constructed from its Fourier coefficients.

Example #3

- Consider the periodic signal:

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad where \quad \omega_0 = \frac{2\pi}{5}$$

- Solution:
  - From the equation of periodicity we can write:

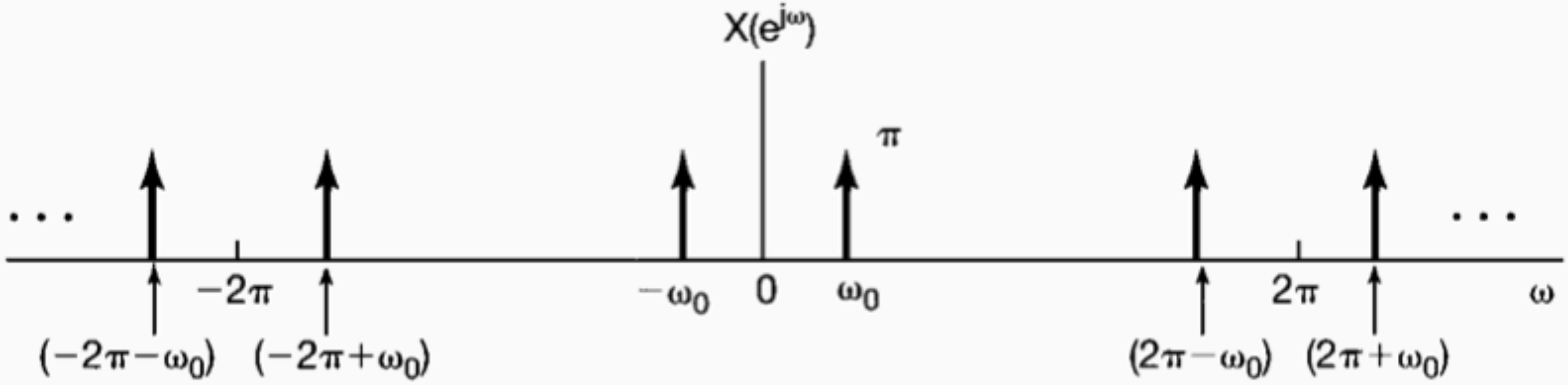
$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} \pi\delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(\omega + \frac{2\pi}{5} - 2\pi l)$$

  - That is,

$$X\left(e^{j\omega}\right) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \le \omega < \pi$$

  - $X(e^{j\omega})$ repeats periodically with a period of 2π, as shown below:

Example #3 (cont.)



Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

Properties of DT Fourier Transform

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Periodicity

- The discrete-time Fourier transform is always periodic in ω with period 2π, i.e.,

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

Linearity

- If:

$$x_1[n] \leftrightarrow X_1\left(e^{j\omega}\right)$$

$$And$$

$$x_2[n] \leftrightarrow X_2\left(e^{j\omega}\right)$$

- Then:

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

Time Shifting & Frequency Shifting

❖ If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

❖ Then:

$$x[n-n_0] \overset{F}{\leftrightarrow} e^{-j\omega_0 n} X\left(e^{j\omega}\right)$$

$$and$$

$$e^{j\omega_0 n} x[n] \overset{F}{\leftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Conjugation & Conjugate Symmetry

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x^*[n] \overset{F}{\leftrightarrow} X^*\left(e^{-j\omega}\right)$$

- If x[n] is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is:

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

- From this, it follows that $\mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$ is an even function of ω and $\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$ is an odd function of ω.
- Similarly the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

Conjugation & Conjugate Symmetry (cont.)

- Furthermore,

$$Ev\{x[n]\} \overset{F}{\leftrightarrow} \mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$$

$$and$$

$$Od\{x[n]\} \overset{F}{\leftrightarrow} j\,\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$$

Differencing & Accumulation

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[n]-x[n-1] \overset{F}{\longleftrightarrow} \left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

$$For \quad signal,$$

$$y[n]=\sum_{m=-\infty}^{n} x[m],$$

- Its Fourier transform is given as:

$$\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{\left(1-e^{-j\omega}\right)} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega-2\pi k)$$

Time Reversal

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[-n] \overset{F}{\longleftrightarrow} X\left(-e^{j\omega}\right)$$

Differentiation in Frequency

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's Relation

❖ If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

❖ Then:

$$\sum_{n=-\infty}^{+\infty} \left|x[n]\right|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Convolution Property

- If x[n], h[n] and y[n] are the input, impulse response, and output respectively, of an LTI system, so that,

$$y[n] = x[n] * h[n]$$

$$then,$$

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$

- Where $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of x[n], h[n] and y[n] respectively.

Example #4

❖ Consider an LTI system with impulse response:

$$h[n] = \delta[n - n_0]$$

❖ The frequency response is:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

❖ Thus for any input x[n] with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is:

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Multiplication Property

❖ It states that:

$$y[n] = x_1[n]x_2[n] \overset{F}{\longleftrightarrow} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\omega}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

Example #5

- Consider the signal:

$$x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$$

- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \delta[n+1]\right)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1]e^{-j\omega n}$$

$$X\left(e^{j\omega}\right) = 1 + e^{-j\omega} + e^{j\omega} = 1 + 2\cos\omega$$

Example #6

- Consider the problem of finding the Fourier transform $X(e^{j\omega})$ of a signal x[n] which is the product of two other signals that is:

$$x[n] = x_1[n]x_2[n]$$

- Where:

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$and$$

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$

- From the multiplication property, we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, where the multiplication integral can be taken over any interval of length 2π. Choosing the interval $-\pi<\theta<\pi$, we obtain:

Example #6 (cont.)

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1\left(e^{j\theta}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

- Above equation resembles aperiodic convolution, except for the fact that the integration is limited to the interval $-\pi < \theta \le \pi$. However, we can convert the equation into an ordinary convolution by defining:

$$\hat{X}_1\left(e^{j\omega}\right)=\begin{cases} X_1\left(e^{j\omega}\right) & for \quad -\pi<\omega\le\pi \\ 0 & otherwise \end{cases}$$

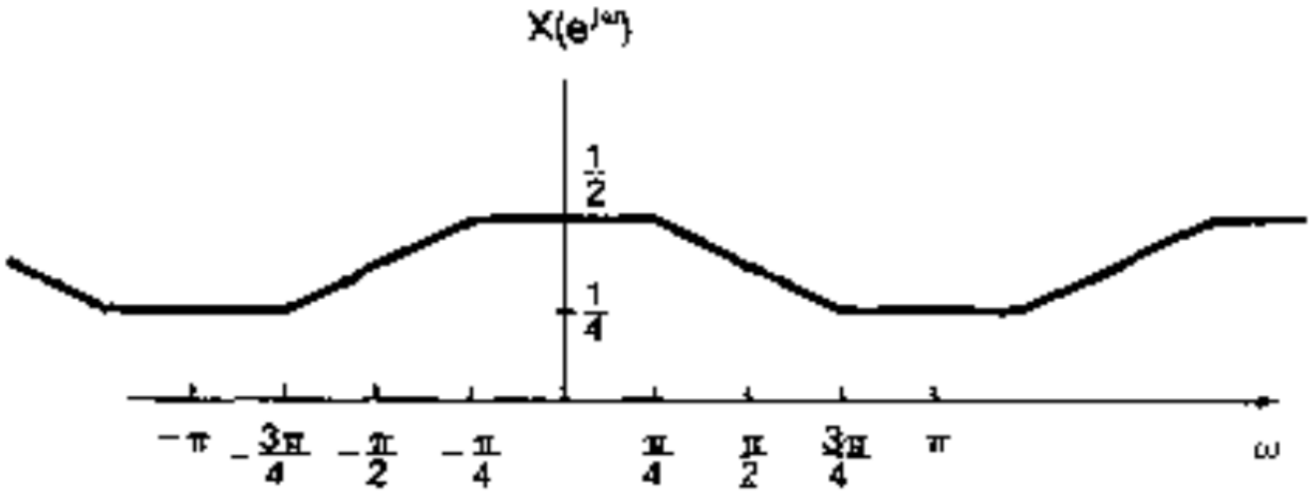
- Then replacing $X_1(e^{j\theta})$ by $X'_1(e^{j\theta})$, and using the fact that $X'_1(e^{j\theta})$ is zero for $|\theta|>\pi$, we see that:

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{X}_1\left(e^{j\theta}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}_1\left(e^{j\theta}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

Example #6 (cont.)

- Thus $X(e^{j\omega})$ is $1/2\pi$ times the aperiodic convolution of the rectangular pulse $X'_1(e^{j\omega})$ and the periodic square wave $X_2(e^{j\omega})$. The result of convolution is the Fourier transform shown below:



Thankyou

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