Signal & Systems

DTFT-II

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Systems Characterized by Linear Constant-Coefficient Difference Equations

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Linear Constant-Coefficient Difference Equations

- A general linear constant-coefficient difference equation for an LTI system with input x[n] and output y[n] is of the form,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Which is usually referred to as Nth-order difference equation.
- If $x[n] = e^{j\omega n}$ is the input to an LTI system, then the output must be of the form $H(e^{j\omega})e^{j\omega n}$. Substituting these expressions into above equation and performing some algebra allow us to solve for $H(e^{j\omega})$.
- Based on convolution, above equation can be written as:

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

Linear Constant-Coefficient Difference Equations (cont.)

- Applying the Fourier transform to both sides and using the linearity and time-shifting properties we obtain the following expression:

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y\left(e^{j\omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\omega} X\left(e^{j\omega}\right)$$

- Or equivalently

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- The frequency response of the LTI system can be written down by inspection as well.

Example #1

- Consider the causal LTI system that is characterized by the difference equation:

$$y[n]-ay[n-1]=x[n],\ \ |a|<1$$

- The frequency response of this system is:

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{1}{1-ae^{-j\omega}}$$

- The impulse response is given by:

$$h[n]=a^{n}u[n]$$

Example #2

- Consider the LTI system:

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

- And let the input to this system be:

$$x[n]=\left(\frac{1}{4}\right)^{n}u[n]$$

- Solution:

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)=\left[\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}$$

Example #2 (cont.)

- Using the partial fraction expansion, we get:

$$Y\left(e^{j\omega}\right)=\frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}}+\frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}+\frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}$$

- Solving the partial fraction gives:

$$B_{11}=-4,\quad B_{12}=-2,\quad B_{21}=8$$

- So that:

$$Y\left(e^{j\omega}\right)=-\frac{4}{1-\frac{1}{4}e^{-j\omega}}-\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}+\frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

- The inverse transform i.e., y[n] is:

$$y[n]=\left\{-4\left(\frac{1}{4}\right)^{n}-2(n+1)\left(\frac{1}{4}\right)^{n}+8\left(\frac{1}{2}\right)^{n}\right\}u[n]$$

Problems

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Problem #1

- Consider the signal:

$$x[n] = \delta[n] + 2\delta[n-1] + 4\delta[n-2]$$

- The impulse response is:

$$h[n] = \delta[n] + \delta[n-1]$$

Problem #2

❖ Use the Fourier transform analysis equation to calculate the Fourier transforms of the following signals:

$$(a):\quad \left(\frac{1}{2}\right)^{|n-1|}$$

$$(b):\quad \delta[n+2]-\delta[n-2]$$

$$(c):\quad \sin\left(\frac{\pi}{2}n\right)+\cos(n)$$

Problem #3

❖ Use the Fourier transform synthesis equation to determine the inverse Fourier transforms of:

$$(a)\quad X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty}\left\{2\pi\delta\left(\omega-2\pi k\right)+\pi\delta\left(\omega-\frac{\pi}{2}-2\pi k\right)+\pi\delta\left(\omega+\frac{\pi}{2}-2\pi k\right)\right\}$$

$$(b)\quad X\left(e^{j\omega}\right) = \cos^2\omega+\sin^2 3\omega$$

Problem #4

- An LTI system with impulse response $h_1[n]=(1/3)^n u[n]$ is connected in parallel with another causal LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response:

$$H\left(e^{j\omega}\right)=\frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-2j\omega}}$$

- Determine $h_2[n]$.

Problem #5

- Consider a causal and stable LTI system S whose input x[n] and output y[n] are related through the second-order difference equation:

$$y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]$$

- (a): Determine the frequency response $H(e^{j\omega})$ for the system S.
- (b): Determine the impulse response h[n] for the system S.

Thankyou

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