Signal & Systems

Sampling

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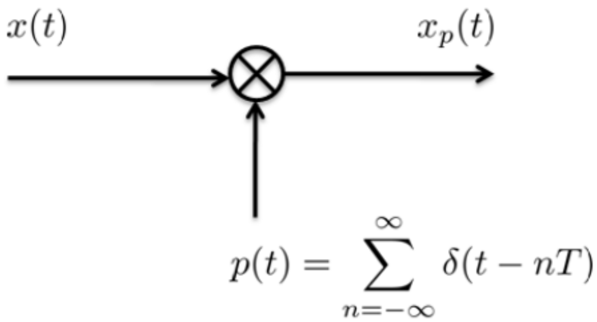
Introduction

- If a signal is band limited i.e., if its Fourier transform is zero outside a finite band of frequencies and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal, then the samples uniquely specify the signal and we can reconstruct it perfectly.
- This result is known as the sampling theorem.
- Sampling theorem plays a crucial role in modern digital signal processing.
- The theorem concerns about the minimum sampling rate required to convert a continuous time signal to a digital signal, without loss of information.

Analog to Digital Conversion

- The analog to digital (A/D) conversion system is shown below:



- The basic idea of A/D conversion is to take a continuous time signal and convert it to a discrete-time signal.
- If the continuous time signal is x(t), we can collect a set of samples multiplying x(t) with an impulse train p(t):

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Where T is the period of the impulse train.

Analog to Digital Conversion (cont.)

- Multiplying x(t) with p(t) yields:

$$x_p(t) = x(t)\,p(t)$$

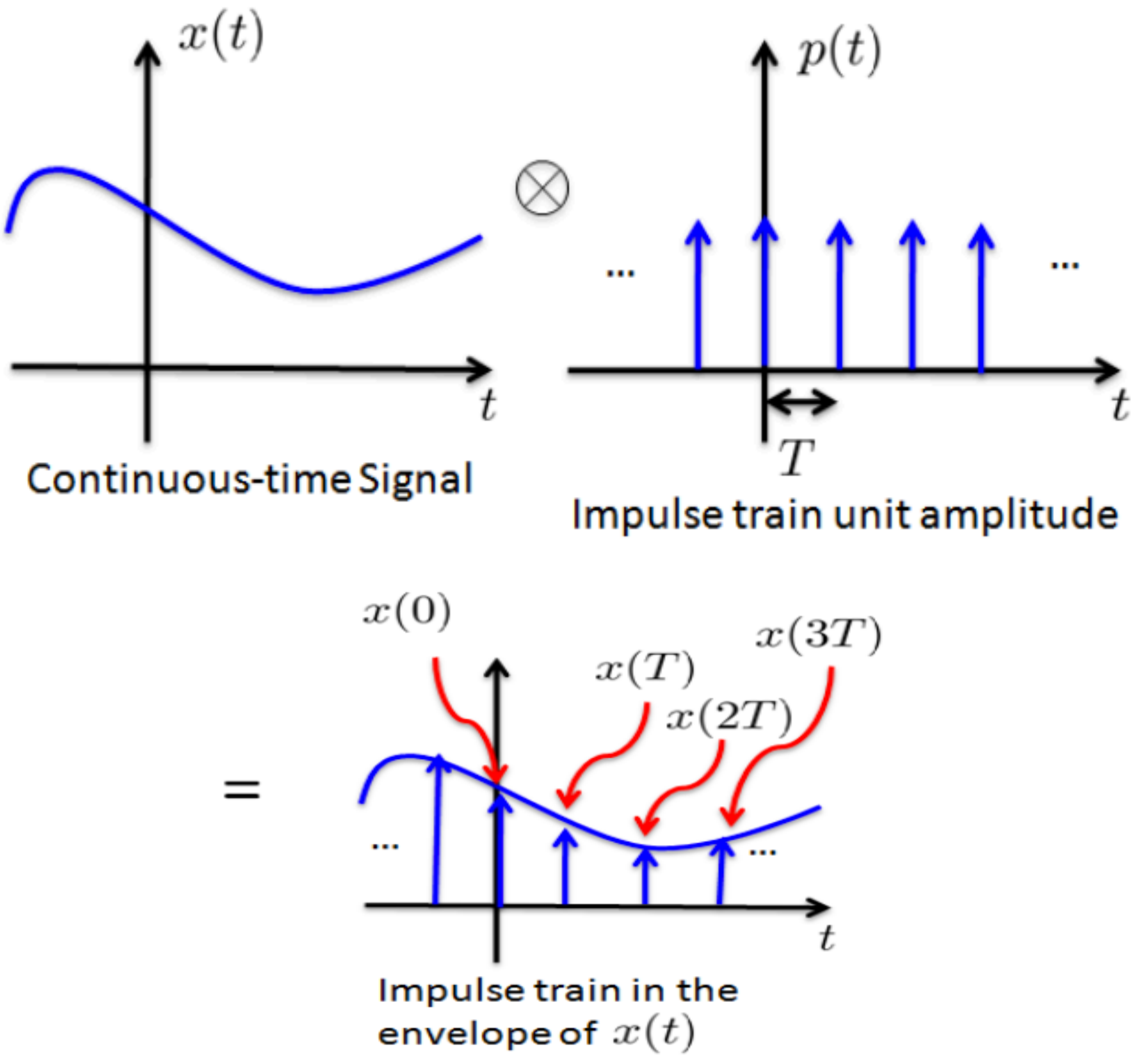
$$= x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty}x(t)\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty}x(nT)\,\delta(t-nT)$$

- Pictorially, $x_p(t)$ is a set of impulses bounded by the envelop x(t) as shown in the next slide:

Analog to Digital Conversion (cont.)



Analog to Digital Conversion (cont.)

- The output signal $x_p(t)$ represents a set of samples of the signal x(t).
- Note that $x_p(t)$ is still a continuous time signal.

Frequency Analysis

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Frequency Analysis of A/D Conversion

- Let us now consider the Fourier transform of $x_p(t)$.
- $x_p(t)$ is the product of $x(t)$ and $p(t)$, the Fourier transform of $x_p(t)$ is the convolution of the Fourier transforms $X(j\omega)$ and $P(j\omega)$.
- From the multiplication property we know that:

$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right]$$

- The Fourier transform of a periodic impulse train is a periodic impulse train.
- Specifically,

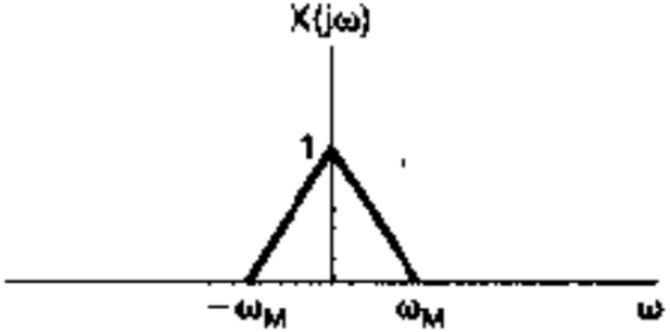
$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

- Since convolution with an impulse simply shifts a signal i.e., $X(j\omega) * \delta(\omega-\omega_0) = X(j(\omega-\omega_0))$, it follows that:

Frequency Analysis of A/D Conversion (cont.)

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\omega_s\right)\right)$$

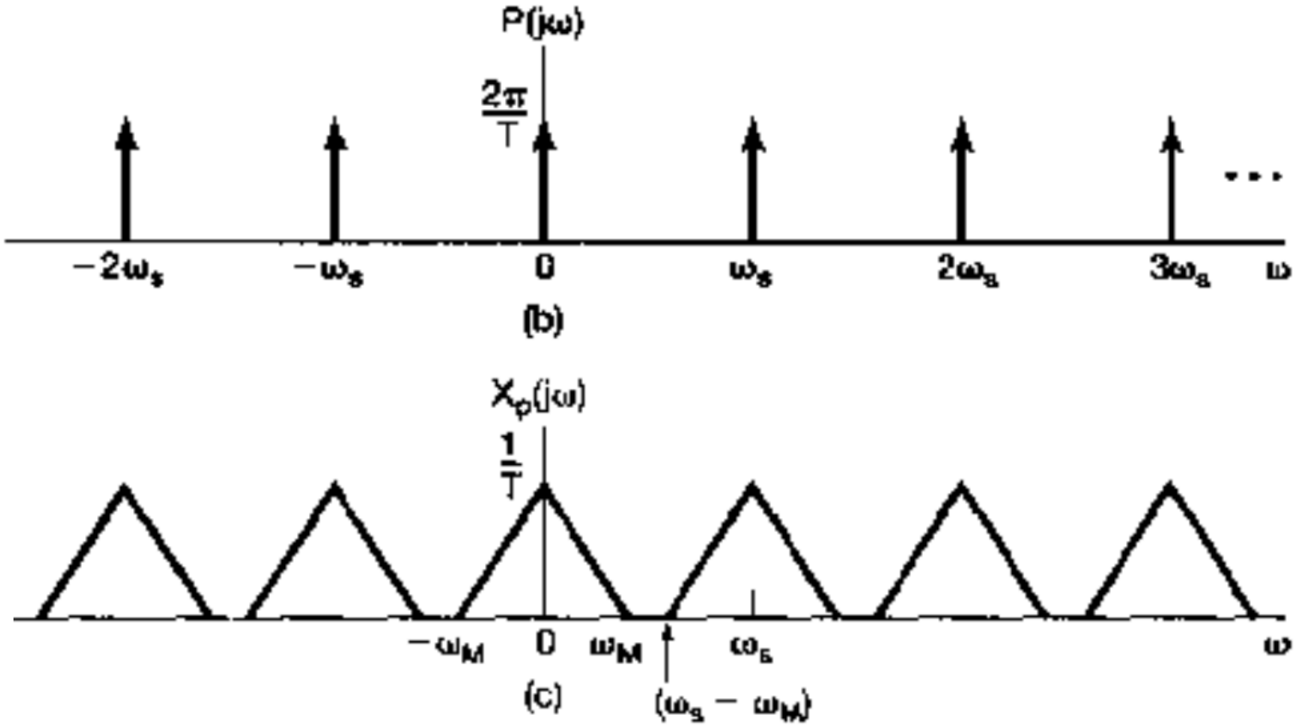
- Above equation provides the relationship between the Fourier transforms of the input and the output of the impulse train modulator.
- $X_p(j\omega)$ is a periodic function of ω consisting of a superposition of shifted replicas of X(jω), scaled by 1/T.
- Illustrated below:



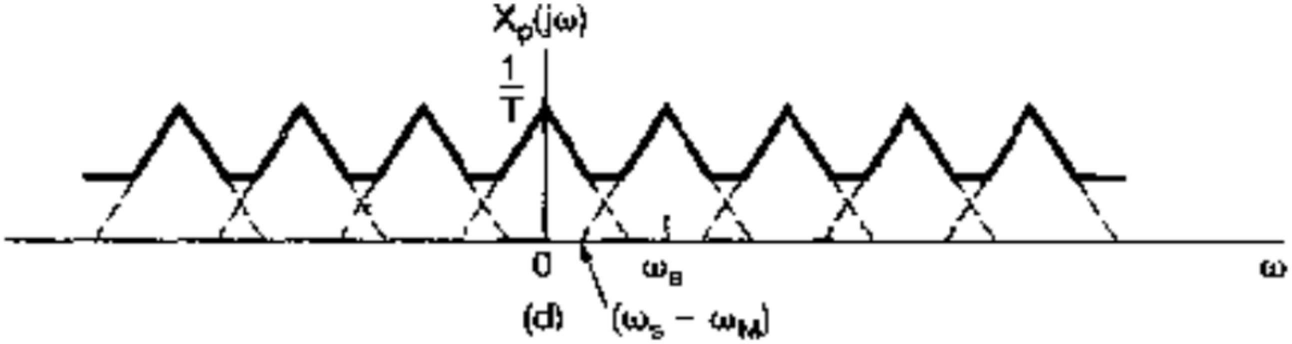
(a)

Frequency Analysis of A/D Conversion (cont.)

- In figure (c) $\omega_M < (\omega_s - \omega_M)$, or equivalently $\omega_s > 2\omega_M$ and thus there is no overlap between the shifted replicas of $X(j\omega)$.
- Whereas in figure (d) with $\omega_s > 2\omega_M$, there is overlap.

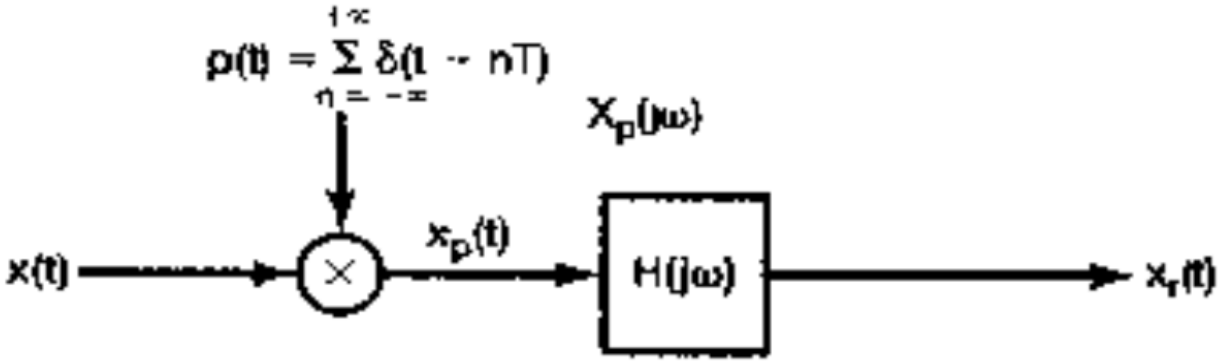


Frequency Analysis of A/D Conversion (cont.)

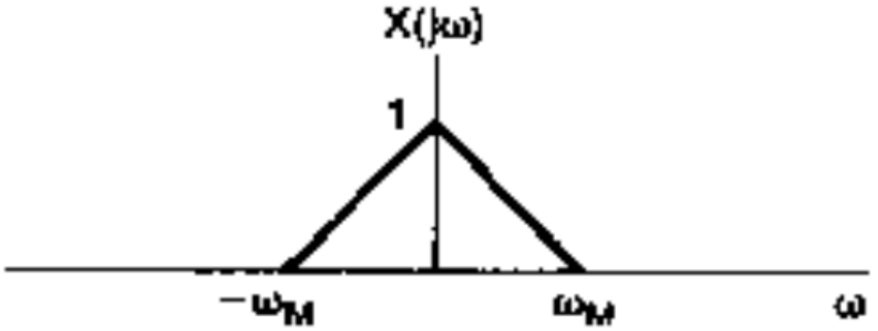


- X (jω) is faithfully reproduced at integer multiples of the sampling frequency.
- If $\omega_s > 2\omega_M$, x(t) can be recovered exactly from $x_p(t)$ by means of a lowass filter with gain T and a cutoff frequency greater than ω_M and less than ω_s- ω_M as depicted below.

Frequency Analysis of A/D Conversion (cont.)

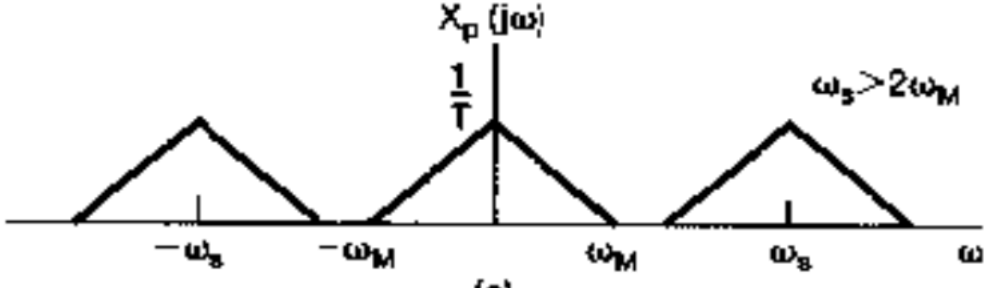


(a)

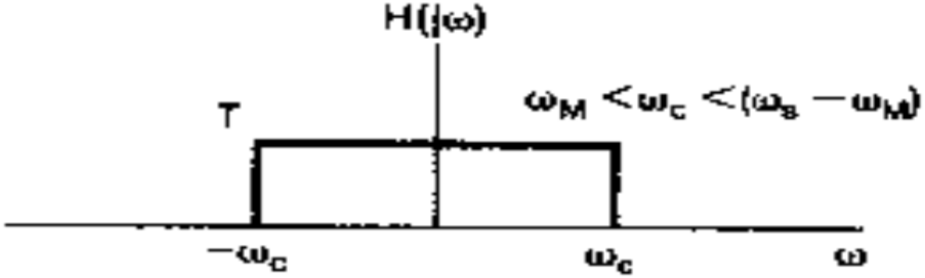


(b)

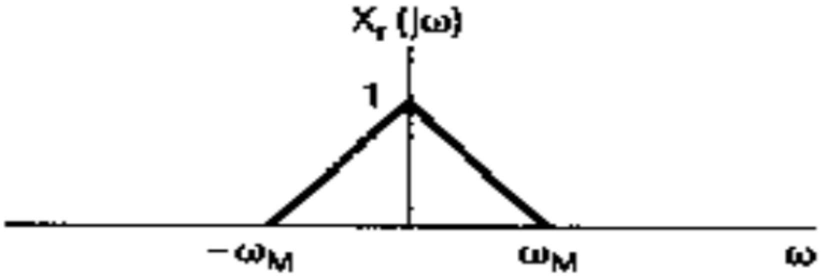
Frequency Analysis of A/D Conversion (cont.)



(c)



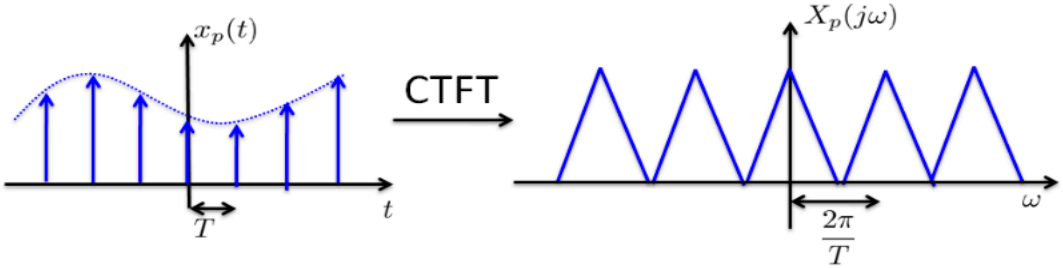
(d)



(e)

What happens if T becomes larger?

- If T becomes larger and larger (i.e., we take fewer and fewer samples), we know from the definition of p(t) that the period in time domain between two consecutive impulses increases farther apart.



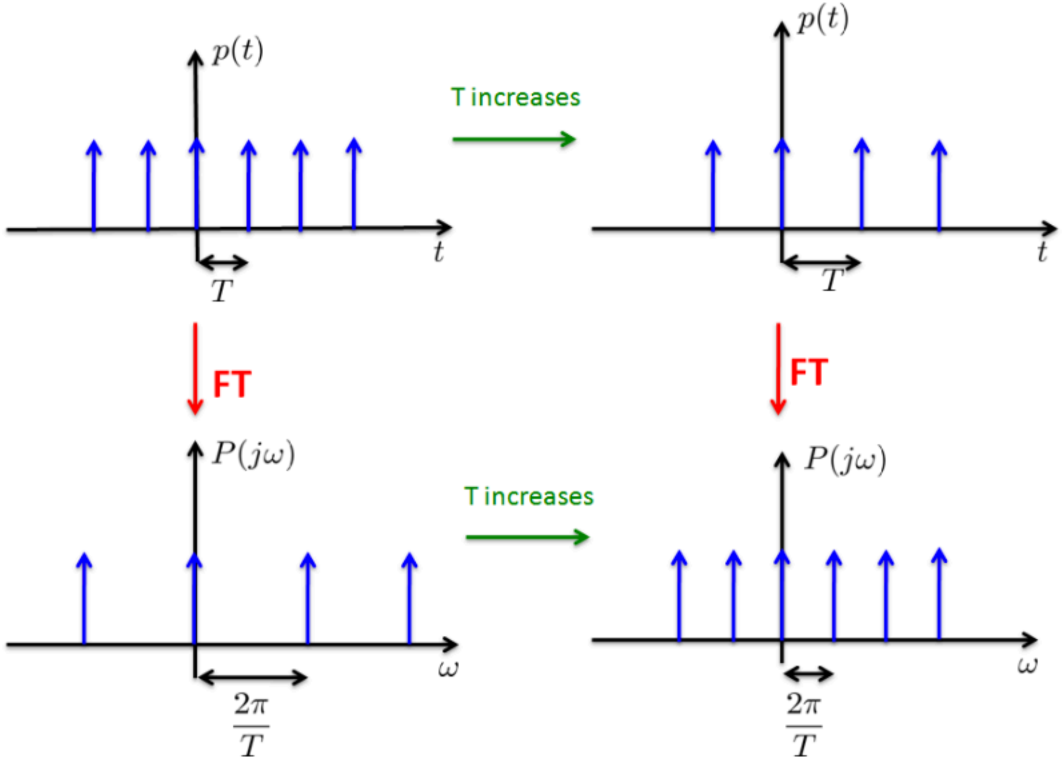
- In frequency domain, since:

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

- The period $2\pi/T$ reduces.
- In other words, the impulses are more packed in frequency domain when T increases.

What happens if T becomes larger?

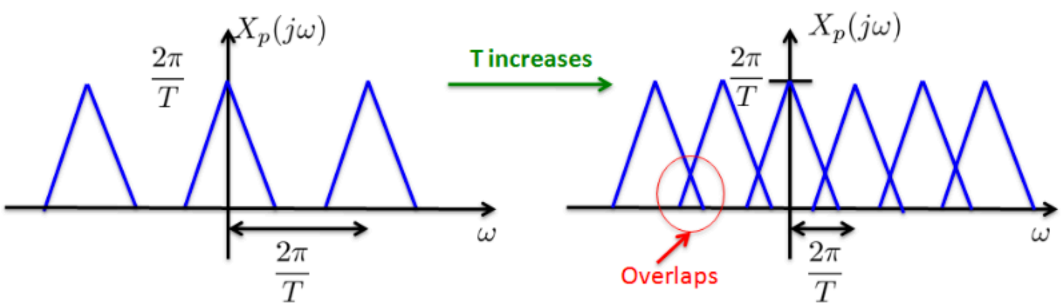
- Figure below illustrates this idea:



- If we consider $X_p(j\omega)$, which is periodic replicate of $X(j\omega)$ at the impulses given by $P(j\omega)$, we see that the seperation between replicates reduces.

What happens if T becomes larger?

- When T hits certain limit, the separation becomes zero and beyond the limit, the replicates start to overlap.
- When the frequency replicates overlap, we say that there is aliasing.

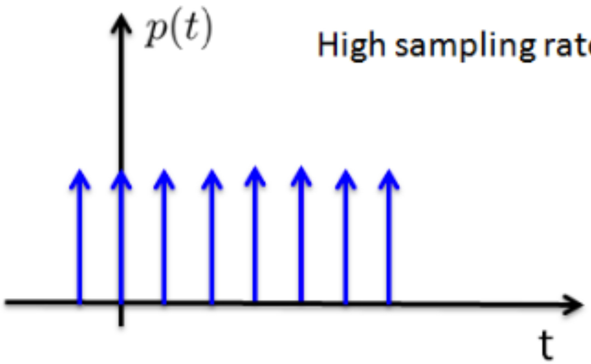


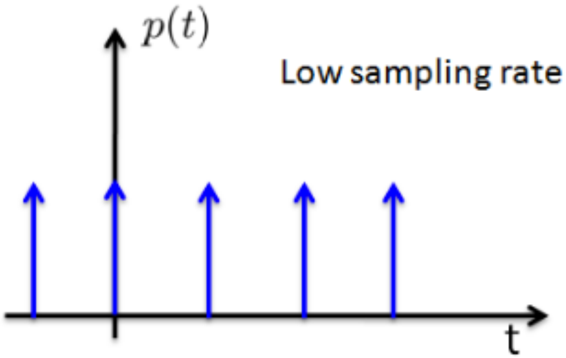
- When T is sufficiently large, there will be overlap between consecutive replicates.
- In order to avoid aliasing T cannot be too large.
- If we define sampling rate to be:

$$\omega_s = \frac{2\pi}{T}$$

What happens if T becomes larger?

❖ Then smaller T implies higher ω_s . In other words there is a minimum sampling rate such that no aliasing occurs.





Sampling Theorem

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Sampling Theorem

- Let x(t) be a band limited signal with X (jω) =0 for all |ω| > W.
- Then the minimum sampling rate such that no aliasing occurs in $X_p(j\omega)$ is:

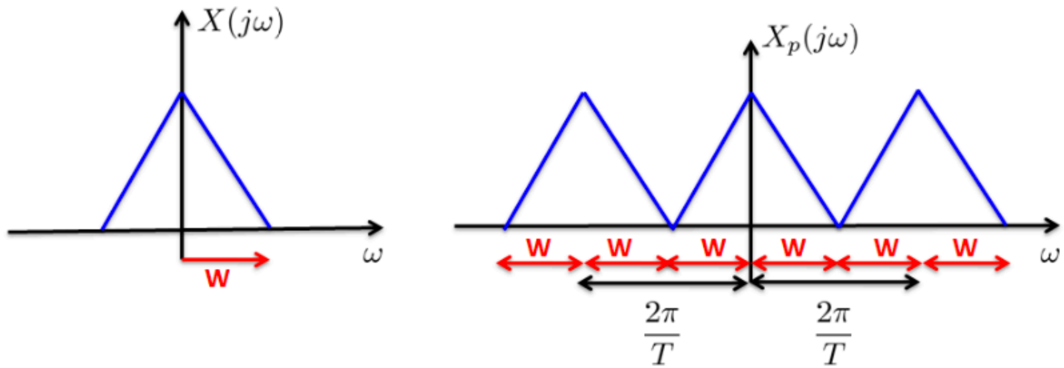
$$\omega_s > 2W$$

- Where

$$\omega_s = \frac{2\pi}{T}$$

Explanation

- Suppose x(t) has bandwidth W.
- The tightest arrangement that no aliasing occurs is shown below:

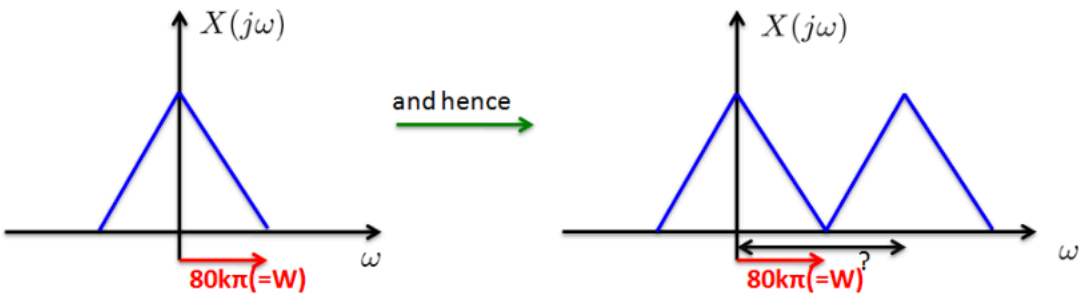


- In this case, we see that the sampling rate ω_s is: ω_s = 2W.
- If T is larger or ω_s is smaller then 2π/T becomes less than 2W and aliasing occurs.
- Therefore the minimum sampling rate to ensure no aliasing is:

$$\omega_s > 2W$$

Example #1

- Suppose there is a signal with maximum frequency 40khz. What is the minimum sampling rate?



- Solution:
- Since, $\omega = 2\pi f$ we know that the max frequency (in rad) is:

$$\omega = 2\pi\left(40\times10^3\right) \Rightarrow 80\times10^3\pi\left(rad\right)$$

- Therefore, the minimum sampling rate is:

$$2\times\left(80\times10^3\pi\right) = 160\times10^3\pi\left(rad\right) \Rightarrow 80kHz$$

Digital to Analog Conversion

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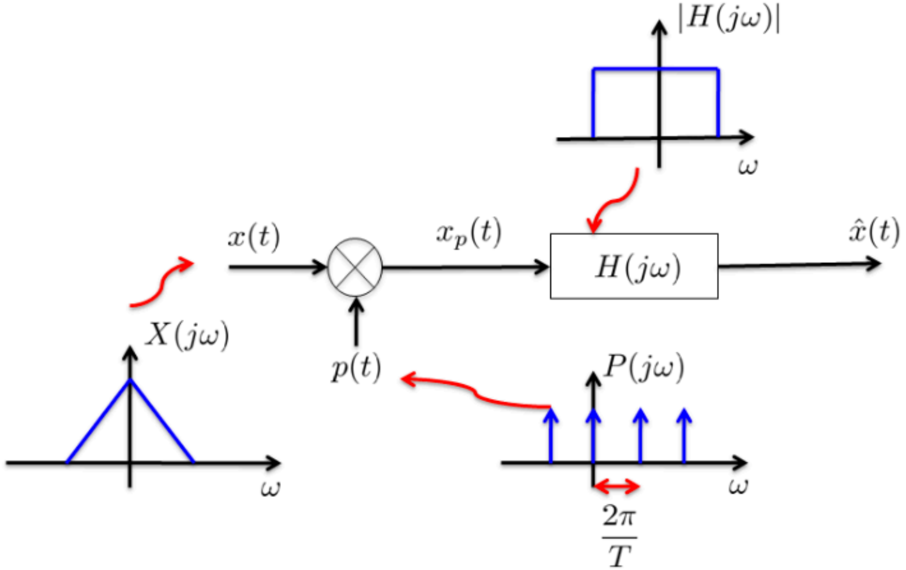
How to recover x (t)?

- If there is no aliasing during the sampling process then we can apply a lowpass filter $H(j\omega)$ to extract the $x(t)$ from $x_p(t)$.
- As shown in a schematic diagram below:



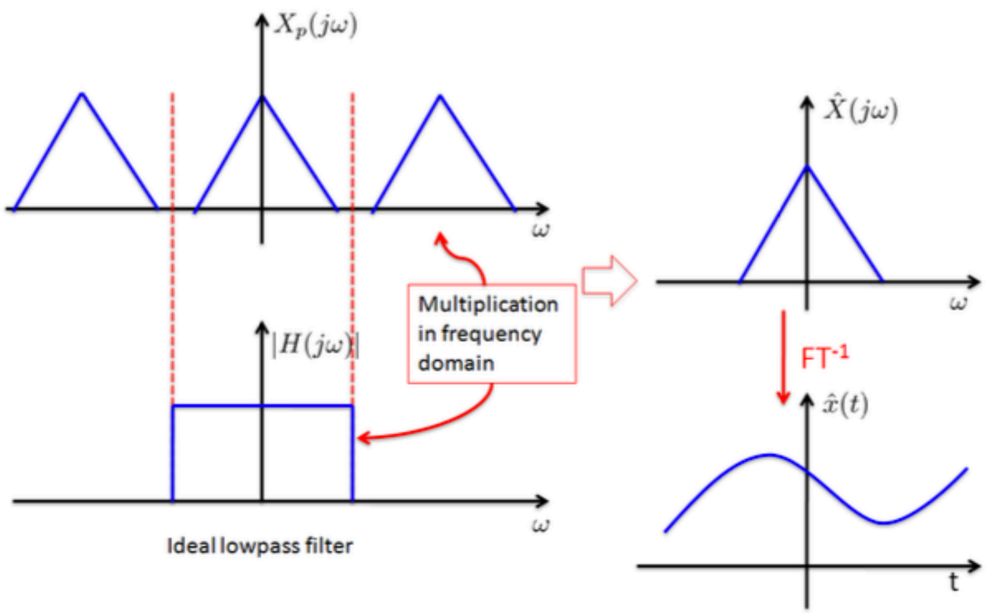
- To see how an ideal lowpass filter can extract $x(t)$ from $x_p(t)$ we first look at the frequency response $X_p\,(j\omega)$.

How to recover x (t)? (cont.)

- Suppose that p(t) has a period of T. Then,

$$X_p(j\omega) = \frac{1}{T}\sum_{-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

- As $X_p(j\omega)$ is a periodic replicate of X (jω). Since we assume that there is no aliasing, the replicate covering the y-axis is identical to X (jω).



How to recover x (t)? (cont.)

- That is for $|\omega| < \omega_s/2$,

$$X_p(j\omega) = X(j\omega)$$

- Now, if we apply an ideal low pass filter:

$$H(j\omega) = \begin{cases} 1, & |\omega| < \dfrac{\omega_s}{2} \\ 0, & otherwise \end{cases}$$

- Then:

$$X_p(j\omega)H(j\omega) = X(j\omega)$$

- For all ω.
- Taking the inverse continuous-time Fourier transform we can obtain x(t).

How to recover x (t)? (cont.)

- If $X_p(t)$ has aliasing, can we still recover x(t) from $x_p(t)$?
- The answer to this question is No.
- As if aliasing occurs, then the condition $X_p(j\omega) = X(j\omega)$ does not hold for all $|\omega| < \omega_s/2$.
- Consequently, even if we apply the lowpass filter $H(j\omega)$ to $X_p(j\omega)$, the result is not $X(j\omega)$.

Thankyou

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