Lecture Notes
21st December 2016

Date WEDNESDAY / 21, DEC, 16 LECTURE # 12

Z-TRANSFORM:-

EXAMPLE # 1:-

$$x[n] = a^n u[n]$$

SOLUTION:-

$$x[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \sum_{-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (a^n z^{-1})^n$$

→ For convergence of $X(z)$, we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$.

→ The region of convergence is the range of values of z for which $|az^{-1}| < 1$ or equivalently $|z| > |a|$

→ Then,

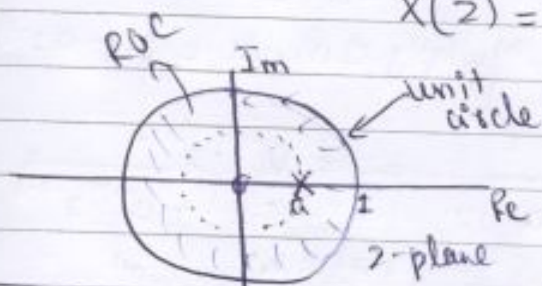
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n \Rightarrow \frac{1}{1 - az^{-1}}$$

$$X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

→ Thus, the z-transform for this signal is well-defined for any value of a, with an ROC determined by the magnitude of a.

→ For example $a = 1$, $x[n]$ is the unit step sequence with z-transform,

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$



Pole zero plot & ROC for $0 < a < 1$.

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→ The z-transform can be characterized by its zeros (the roots of the numerator polynomial) and its poles (the roots of the denominator polynomial).

$$X(z) = \frac{N \rightarrow (\text{zeros})}{D \rightarrow (\text{poles})}$$

⇒ For previous example #1 there is one zero at $z_1 = 0$ and one pole at $z = a$.

⇒ If $a > 1$, then the ROC doesnot include the unit circle, consistent with the fact that, for those values of a, the Fourier transform of $a^n u[n]$ does not converge.

EXAMPLE #2:-

$$x[n] = -a^n u[-n-1] \text{ with } 0 < a < 1.$$

SOL:-

$$x[n] = \begin{cases} -a^n u[-n-1] & \text{for } n \leq 0 \\ 0 & \text{for } n > 0 \end{cases}$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

→ If $|a^{-1} z| < 1$ or equivalently $|z| < |a|$, the above sum converges,

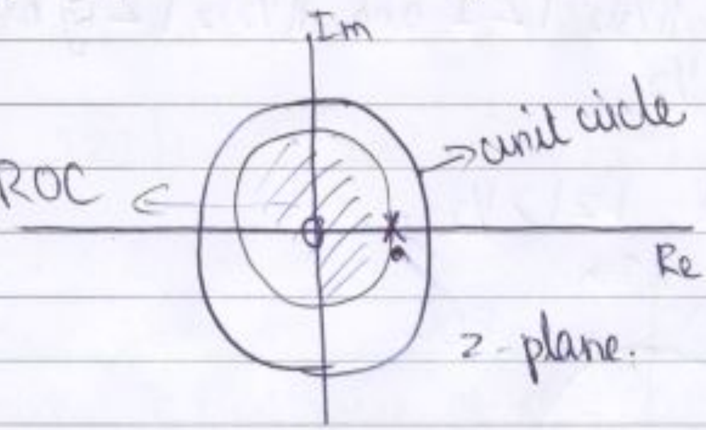
$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{\cancel{1} - a^{-1} z - \cancel{1}}{1 - a^{-1} z}$$

$$= \frac{-a^{-1} z}{1 - a^{-1} z} \quad \text{Multiply \& divide with } (-az^{-1})$$

$$= \frac{(-az^{-1})(-a^{-1} z)}{(-az^{-1}) + (az^{-1})(a^{-1})z} = \frac{+a^{-1+1} z^{-1+1}}{-az^{-1} + a^{-1+1} z^{-1+1}} = \frac{a^0 z^0}{-az^{-1} + a^0 z^0}$$

$$X(z) = \frac{1}{1 - az^{-1}} \Rightarrow \frac{z}{z-a}, \quad |z| < |a|$$

→ The pole-zero plot and ROC for a value of a between 0 & 1 is:-



EXAMPLE # 3:-

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$1 - \frac{3}{2} z^{-1}$

$z - \frac{3}{2}$

SOL:-

The z-transform of $x(n)$ is:-

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n) \right\} z^{-n}$$

$$= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n} - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}$$

$$= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

$$= 7\left[\frac{1}{1 - \frac{1}{3}z^{-1}}\right] - 6\left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right]$$

$$= \frac{7}{1 - \frac{1}{3}z^{-1}} - \frac{6}{1 - \frac{1}{2}z^{-1}} = \frac{7\left(1 - \frac{1}{2}z^{-1}\right) - 6\left(1 - \frac{1}{3}z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

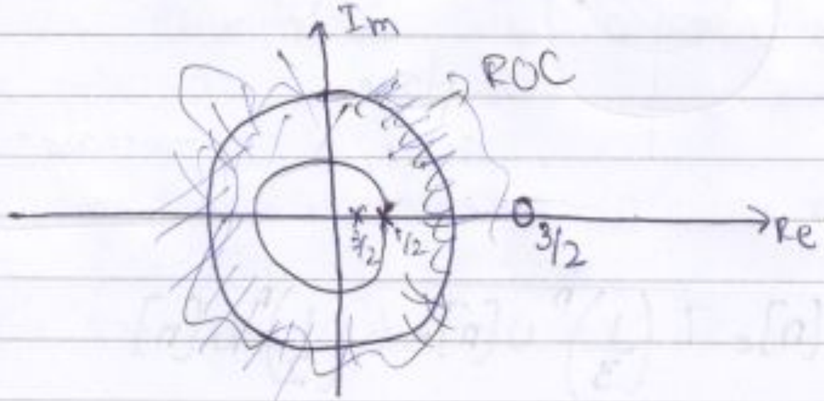
$$X(z) = \frac{7 - \frac{7}{2}z^{-1} - 6 + \frac{6}{3}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} = \frac{1 - \frac{3}{2}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$X(z) = \frac{z\left(z - \frac{3}{2}\right)}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

→ For convergence of $X(z)$, both sums must converge.

→ So we require both $|(1/3)z^{-1}| < 1$ and $|(1/2)z^{-1}| < 1$ or equivalently $|z| > 1/3$ & $|z| > 1/2$.

→ Thus the ROC is $|z| > 1/2$.



EXAMPLE # 4:-

Find the convolution of the sequences:-

$$x_1 = \{\underset{\uparrow}{1}, -3, 2\} \ \& \ x_2 = \{\underset{\uparrow}{1}, 2, 1\}$$

Sol:-

STEP#1:- Determine the z-transform of individual signal sequences.

$$X_1(z) = Z\{x_1[n]\} = \sum_{n=0}^{2} x_1(n) z^{-n}$$

$$= x_1(0)z^{-0} + x_1(1)z^{-1} + x_2(2)z^{-2}$$

$$X_1(z) = 1z^{0} + (-3)z^{-1} + 2z^{-2} \Rightarrow 1 - 3z^{-1} + 2z^{-2}$$

$$X_2(z) = Z\{x_2[n]\} = \sum_{n=0}^{2} x_2[n] z^{-n}$$

$$= x_2(0)z^{0} + x_2(1)z^{-1} + x_2(2)z^{-2}$$

$$X_2(z) = 1z^{0} + 2z^{-1} + 1z^{-2} \Rightarrow 1 + 2z^{-1} + z^{-2}$$

STEP#2:- Multiplication of $X_1(z)$ & $X_2(z)$:

$$X(z) = X_1(z)\,X_2(z) = (1 - 3z^{-1} + 2z^{-2})(1 + 2z^{-1} + z^{-2})$$

$$= 1 + 2z^{-1} + z^{-2} - 3z^{-1} - 6z^{-2} - 3z^{-3} + 2z^{-2} + 4z^{-3} + 2z^{-4}$$

$$X(z) \Rightarrow 1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}$$

STEP#3:- Let us take inverse z-transform of $X(z)$:

$$x[n] = \text{IZT}\left[1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}\right] \Rightarrow \{1, -1, -3, 1, 2\}$$

EXAMPLE # 5:-

Find the initial & Final value of $x[n]$ if its z-transform $X(z)$ is given by:-

$$X(z) = \frac{0.5z^2}{(z-1)(z^2 - 0.85z + 0.35)}$$

SOL:-

$\Rightarrow$ The initial value of $x(0)$ is given by:-

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{0.5z^2}{(z-1)(z^2 - 0.85z + 0.35)}$$

$$= \lim_{z\to\infty} \frac{0.5z^2}{z^3 - 0.85z^2 + 0.35z - z^2 + 0.85z - 0.35}$$

$$x(0) = \lim_{z\to\infty} \frac{0.5z^2}{z(z^2)} \Rightarrow \lim_{z\to\infty} \frac{0.5}{z} \Rightarrow 0.$$

$\Rightarrow$ The final value or steady value of $x[n]$ is given by:-

$$x(\infty) = \lim_{z\to 1} \left[(z-1)X(z)\right] = \lim_{z\to 1} \cancel{(z-1)} \frac{0.5z^2}{\cancel{(z-1)}(z^2 - 0.85z + 0.35)}$$

$$x(\infty) = \lim_{z\to 1} \frac{0.5z^2}{z^2 - 0.85z + 0.35} \Rightarrow \frac{0.5}{(1 - 0.85 + 0.35)} \Rightarrow \frac{0.5}{0.5}$$

$$x(\infty) \Rightarrow 1$$

EXAMPLE #6:-

$$h[n] = \delta[n] + \delta[n-1] + 2\delta[n-2].$$

$$H(z) = ?$$

SOL:-

Sequence Z-T ROC

$\because \delta[n] \Rightarrow 1$ (all z in ROC)

$\because \delta[n-m] \Rightarrow z^{-m}$ (all z except 0 (if $m>0$) or ∞ (if $m<0$))

The z-transform of $h[n]$ is:-

$$H(z) \Rightarrow 1 + z^{-1} + 2z^{-2}$$

EXAMPLE #7:-

$$x[n] = \left(\frac{1}{3}\right)\sin\left(\frac{\pi}{4}n\right)u[n]$$

$$X(z) = ?$$

SOL:-

$$x[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=0}^{\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n\right\}z^{-n} - \sum_{n=0}^{\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n\right\}z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{j\pi/4}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\pi/4}z^{-1}\right)^n$$

$$X(z) = \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}$$

$$= \frac{\left(1-\frac{1}{3}e^{-j\pi/4}z^{-1}\right) - \left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)}{2j\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-\frac{1}{3}e^{-j\pi/4}z^{-1}\right)}$$

$$= \frac{\cancel{1}-\frac{1}{3}e^{-j\pi/4}z^{-1} - \cancel{1} + 1/3\,e^{j\pi/4}z^{-1}}{2j\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-1/3\,e^{-j\pi/4}z^{-1}\right)}$$

$$= \frac{-\frac{1}{3}e^{j\pi/4}z^{-1} + \frac{1}{3}e^{j\pi/4}z^{-1}}{2j\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-\frac{1}{3}e^{-j\pi/4}z^{-1}\right)}$$

$$= \frac{1/3\, z^{-1}\left(-e^{-j\pi/4} + e^{j\pi/4}\right)/2j}{\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)}$$

$$= \frac{1/3\, z^{-1}\left(\sin\pi/4\right)}{\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-\frac{1}{3}e^{-j\pi/4}z^{-1}\right)}$$

$$X(z) = \frac{1/3\, z^{-1}\,(0.70710)}{\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-\frac{1}{3}e^{-j\pi/4}z^{-1}\right)} \Rightarrow \frac{\left(1/3\sqrt{2}\right)z^{-1}}{\left(1-\frac{1}{3}e^{j\pi/4}z^{-1}\right)\left(1-\frac{1}{3}e^{-j\pi/4}z^{-1}\right)}$$