Signal & Systems

Z-Transform-I

21st December 16

Z-Transform

21^{st} December 16

The Z-Transform

- The z-transform of a discrete-time signal x[n] is:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- The z-transform operation is denoted as:

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

- Where "z" is the complex number. Therefore, we may write z as:

$$z = re^{j\omega}$$

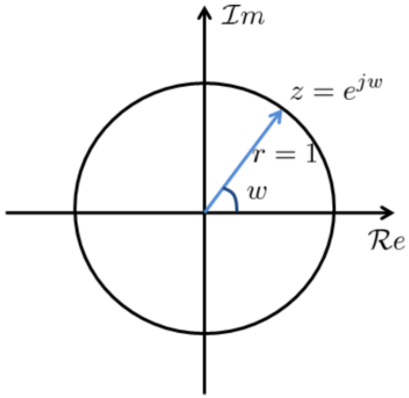
- Where r and ω belongs to Real number. When r=-1, the z-transform of a discrete-time signal becomes:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- Therefore, the DTFT is a special case of the z-transform.

The Z-Transform (cont.)

- Pictorially, we can view DTFT as the z-transform evaluated on the unit circle:



- When r≠1, the z-transform is equivalent to:

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(r^{-n}x[n]\right)e^{-j\omega n}$$

$$= F\left[r^{-n}x[n]\right]$$

The Z-Transform (cont.)

- Which is the DTFT of the signal $r^{-n} x[n]$.
- However, we know that the DTFT does not always exist. It exists only when the signal is square summable, or satisfies the Dirichlet conditions.
- Therefore, X(z) does not always converge. It converges only for some values of r. this range of r is called the region of convergence (ROC).

Region of Convergence (ROC)

- The region of convergence are the values of for which the z-transform converges.
- Z-transform is an infinite power series which is not always convergent for all values of z.
- Therefore, the region of convergence should be mentioned along with the z-transformation.
- The Region of Convergence (ROC) of the z-transform is the set of z such that X(z) converges, i.e.,

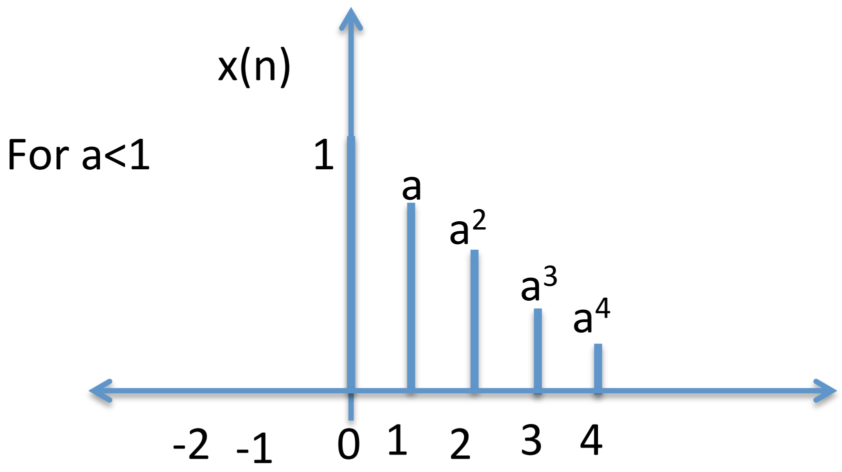
$$\sum_{n=-\infty}^{\infty} \left| x[n] \right| r^{-n} < \infty$$

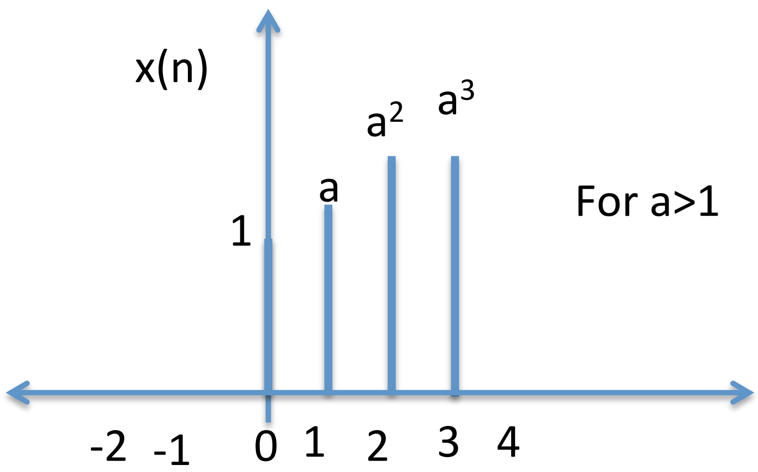
Example #1

- Z-transform of right-sided exponential sequences.
- Consider the signal $x[n] = a^n u[n]$. Mathematically it can be written as:

$$x(n) = \begin{cases} a^n & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$

- This sequence exists for positive values of n:





Example #1 (cont.)

- The z-transform of x[n] is given by:

$$X(z)=\sum_{-\infty}^{\infty}a^n u[n]z^{-n}$$

$$=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

- Therefore, X(z) converges if $\sum_{n=0}^{\infty}\left(az^{-1}\right)^n<\infty$. From geometric series, we know that:

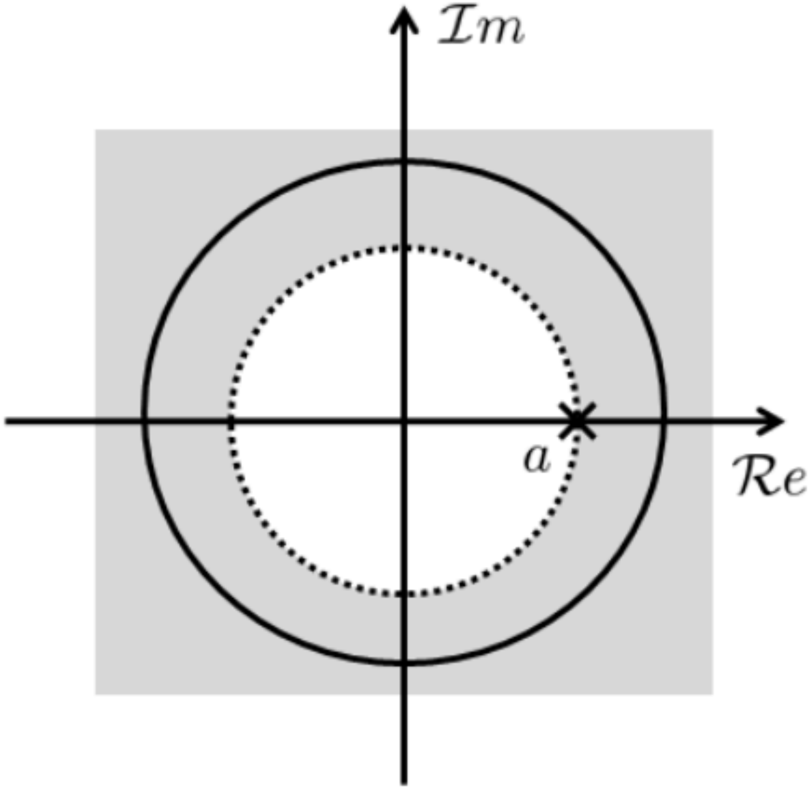
$$\sum_{n=0}^{\infty}\left(az^{-1}\right)^n=\frac{1}{1-az^{-1}}$$

- When $|az^{-1}|<1$, or equivalently $|z| > |a|$. So,

$$X(z)=\frac{1}{1-az^{-1}}$$

Example #1 (cont.)

❖ With ROC being the set of z such that |z| > |a|. As shown below:

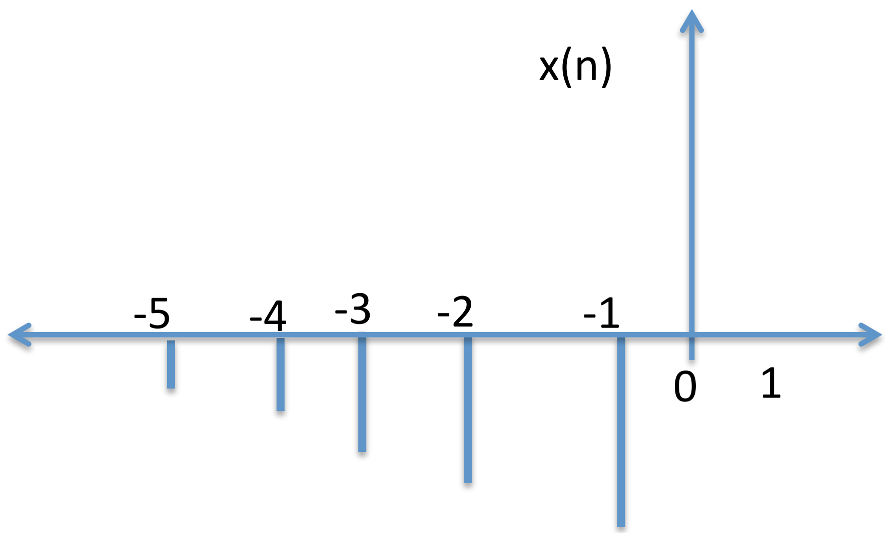


Example #2

- Consider the signal $x[n] = -a^n u[-n-1]$ with $0 < a < 1$. Which is the left sided exponential sequence and it can be determined mathematically as:

$$x(n) = \begin{cases} -a^n u(-n-1) & for \quad n \le 0 \\ 0 & for \quad n > 0 \end{cases}$$

- This sequence exists only for negative values of n.



Example #2 (cont.)

- The z-transform of x[n] is: $X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

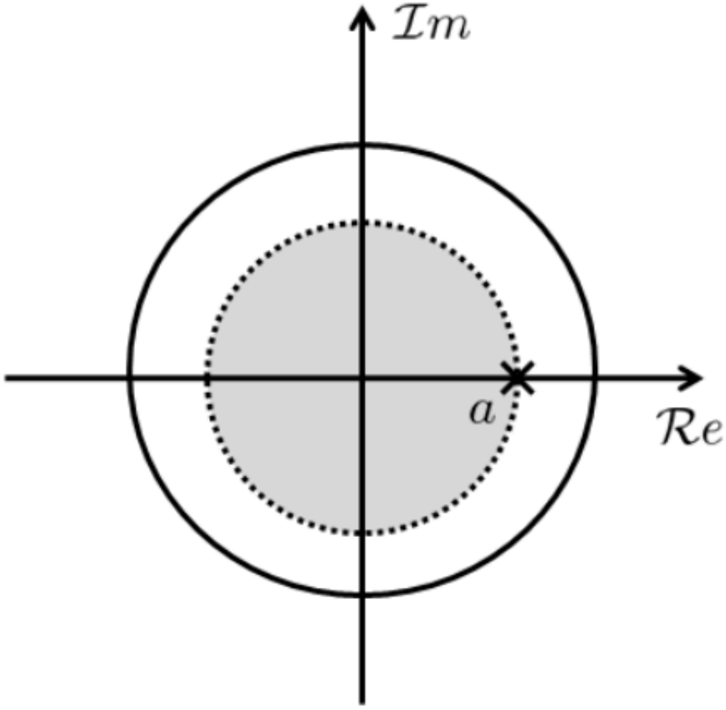
$$= 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n$$

- Therefore, X(z) converges when $|a^{-1}z| < 1$, or equivalently $|z| < |a|$. In this case:

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

Example #2 (cont.)

- Witch ROC being the set of z such that |z| < |a|.
- Note that the z-transform is the same as that of Example 1, the only difference is the ROC. Which is shown below as:



Example #3

- Consider the signal: $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$
- The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right]u[n]z^{-n}$$

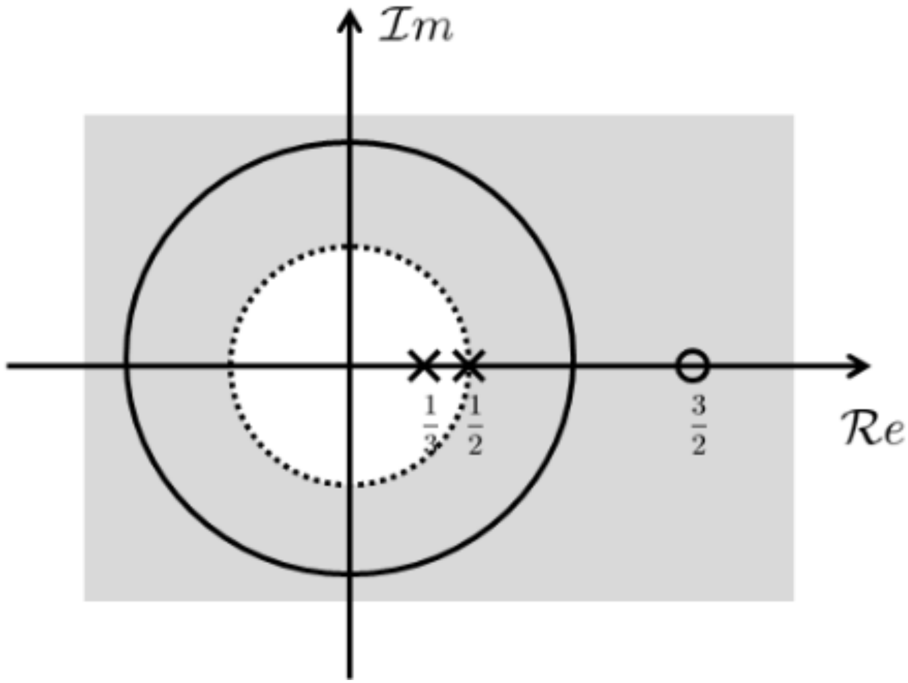
$$= 7\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^n u[n]z^{-n} - 6\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n}$$

$$= 7\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) - 6\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

$$= \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

Example #3 (cont.)

- For X(z) to converge, both sums in X(z) must converge.
- So we need both |z| > |1/3| and |z| > |1/2|. Thus the ROC is the set of z such that |z| > |1/2|.



Properties of Z-Transform

21st December 16

Linearity

- This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$,then

$$a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

- Where a_1 and a_2 are constants.

Time Scaling

- This property of z-transform states that if $x(n) \overset{z}{\longleftrightarrow} X(z)$,then we can write:

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

- Where k is an integer which is shift in time in x(n) in samples.

Scaling in Z-Domain

- This property states that if:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad ROC: r_1 < |z| < r_2$$

$$then \quad a^n x(n) \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right) \quad ROC: |a| r_1 < |z| < |a| r_2$$

- Where a is a constant.

Time Reversal

- This property states that if:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \qquad ROC: r_1 < |z| < r_2$$

- Then,

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) \quad ROC: \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Differentiation in Z-domain

- This property states that if $x(n) \overset{z}{\longleftrightarrow} X(z)$, then:

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{d\{X(z)\}}{dz}$$

Convolution

❖ This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then:

$$x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z)X_2(z)$$

Example #4

- Find the convolution of sequences:

$$x_1 = \{1, -3, 2\} \quad and \quad x_2 = \{1, 2, 1\}$$

- Solution:
- Step 1: Determine z-transform of individual signal sequences:

$$X_1(z) = Z\left[x_1(n)\right] = \sum_{n=0}^{2} x_1(n) z^{-n} = x_1(0)z^0 + x_1(1)z^{-1} + x_1(2)z^{-2}$$

$$= 1z^0 - 3z^{-1} + 2z^{-2} = 1 - 3z^{-1} + 2z^{-2}$$

$$and \quad X_2(z) = Z\left[x_2(n)\right] = \sum_{n=0}^{2} x_2(n) z^{-n} = x_2(0)z^0 + x_2(1)z^{-1} + x_2(2)z^{-2}$$

$$= 1z^0 + 2z^{-1} + 1z^{-2} = 1 + 2z^{-1} + 1z^{-2}$$

Example #4 (cont.)

- Step 2: Multiplication of $X_1(z)$ and $X_2(z)$:

$$X(z) = X_1(z)X_2(z) = \left(1 - 3z^{-1} + 2z^{-2}\right)\left(1 + 2z^{-1} + 1z^{-2}\right)$$

$$= 1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}$$

- Step 3: Let us take inverse z-transform of X(z):

$$x(n) = IZT\left[1 - z^{-1} - 3z^{-2} + z^{-3} + 2z^{-4}\right] = \{1, -1, -3, 1, 2\}$$

Other Properties of Z-Transform

- **Correlation of Two Sequences:**
  - This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then,

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2(n-m) \overset{z}{\longleftrightarrow} X_1(z)X_2(z^{-1})$$

- **Multiplication:**
  - This property states that if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then:

$$x_1(n)x_2(n) \overset{z}{\longleftrightarrow} \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{z}{v}\right) z^{-1}\, dv$$

  - Where C is the closed contour which encloses the origin and lies in the ROC that is common to both X1(n) and X1(v) and X2 (1/v).

Other Properties of Z-Transform (cont.)

❖ Conjugate of a Complex sequence:

- ❖ This property of z-transform states that if x(n) is a complex sequence and if $x(n) \stackrel{z}{\longleftrightarrow} X(z)$ then,

$$x^*(n) \stackrel{z}{\longleftrightarrow} \left[X(z^*)\right]^*$$

❖ Real part of a sequence:

- ❖ This property of z-transform states that if x(n) is a complex sequence and if $x(n) \stackrel{z}{\longleftrightarrow} X(z)$ then,

$$\operatorname{Re}\left[x(n)\right] \stackrel{z}{\longleftrightarrow} \frac{1}{2}\left[X(z) + X^*(z^*)\right]$$

Other Properties of Z-Transform (cont.)

- **Imaginary part of a sequence:**
  - This property of z-transform states that if x(n) is a complex sequence and if $x(n) \overset{z}{\longleftrightarrow} X(z)$ then,

$$\operatorname{Im}\left[x(n)\right] \overset{z}{\longleftrightarrow} \frac{1}{2j}\left[X(z) - X^{*}(z^{*})\right]$$

- **Initial value theorem:**
  - The initial value theorem foe a single-sided sequence x(n) is x(0) whereas this is x(-∞) for double-sided sequence.
  - It is difficult to get this value from the knowledge of one-sided z-transform.
  - This theorem states that for a casual sequence x(n), x(0) can be obtained by the knowledge ox X(z) i.e., one-sided of x(z) i.e.,

$$x(0) = \underset{z \to \infty}{LT} X(z)$$

Other Properties of Z-Transform (cont.)

- **Final value theorem:**
  - The final value theorem states that if a sequence x(n) has finite value as n\∞, called as x(∞), then this value can be determined by the knowledge of its one-sided z-transform i.e.,

$$\underset{n\to\infty}{LT}\, x(n) = x(\infty)\,\underset{z\to 1}{LT}\left[(z-1)X(z)\right]$$

- **Partial Sum:**
  - It states that:

$$\sum_{-\infty}^{\infty} x(n) \leftrightarrow \frac{X(z)}{1-z^{-1}}$$

Other Properties of Z-Transform (cont.)

- **Parseval's Theorem:**
  - It states that:

$$\sum_{n=-\infty}^{\infty} x_1(n)\,x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)\,X_2^*\left(\frac{1}{v}\right)v^{-1}\,dv$$

  - Where $x_1(n)$ and $x_2(n)$ are complex valued sequences.

Example #5

- Find the initial and final value of x(n) if its z-transform X(z) is given by:

$$X(z) = \frac{0.5z^2}{(z-1)(z^2-0.85z+0.35)}$$

- Solution:
  - The initial value x(0) is given by:

$$x(0) = \underset{z\to\infty}{LT} X(z) = \underset{z\to\infty}{LT} \frac{0.5z^2}{(z-1)(z^2-0.85z+0.35)} = \underset{z\to\infty}{LT} \frac{0.5z^2}{(z)(z^2)} = 0$$

  - The final value or steady value of x(n) is given by:

$$x(\infty) = \underset{z\to 1}{LT}\left[(z-1)X(z)\right] = \frac{0.5}{(1-0.85+0.35)} = 1.0$$

Standard Z-transform Pairs

21st December 16

Z-Transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$

Z-Transform Pairs (cont.)

7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Example #6

- Consider the signal $h[n] = \delta[n] + \delta[n-1] + 2\delta[n-2]$ .Find the z-transform of h[n].
- Solution:
  - The z-transform of h[n] is:

$$H(z) = 1 + z^{-1} + 2z^{-2}$$

Example #7

- Consider the signal.

$$x[n] = \left(\frac{1}{3}\right)\sin\left(\frac{\pi}{4}n\right)u[n]$$

- Find the Z-Transform.

Thankyou

21st December 16