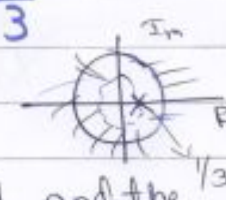
Lecture Notes
23rd December 2016

23rd DEC/FRIDAY — LECTURE #13

EXAMPLE #1 :-

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \quad , \quad |z| > \frac{1}{3}$$



SOL:-

→ There are two poles, one at $z=\frac{1}{3}$ and one at $z=\frac{1}{4}$ and the ROC lies outside the outermost pole.

→ To solve let's expand $X(z)$ using partial fractions.

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}} \Rightarrow ①$$

cross multiplication gives,

$$X(z) = 3-\frac{5}{6}z^{-1} = A\left(1-\frac{1}{3}z^{-1}\right) + B\left(1-\frac{1}{4}z^{-1}\right) \rightarrow ②$$

$$-\frac{1}{3}z^{-1} = -1 \quad , \quad -\frac{1}{4}z^{-1} = -1$$

$$z^{-1} = 3 \quad , \quad z^{-1} = 4$$

Put $z^{-1}=3$ in equ ②

$$3-\frac{5}{6}(3) = A\left[1-\frac{1}{3}(3)\right] + B\left[1-\frac{1}{4}(3)\right]$$

$$3-\frac{5}{2} = A[1-1] + B\left[1-\frac{3}{4}\right]$$

$$\frac{6-5}{2} = 0 + B\left[\frac{4-3}{4}\right]$$

$$\frac{1}{2} = B\left[\frac{1}{4}\right] \quad , \quad B = \frac{1}{2}\times 4 \Rightarrow 2.$$

Now put $z^{-1}=4$ in equ ②

$$3-\frac{5}{6}(4) = A\left[1-\frac{1}{3}(4)\right] + B\left[1-\frac{1}{4}(4)\right]$$

$$3-\frac{10}{3} = A\left[1-\frac{4}{3}\right] + 0$$

$$\frac{9-10}{3} = A\left[\frac{3-4}{3}\right]$$

$$-\frac{1}{3} = A\left[-\frac{1}{3}\right] \quad , \quad A = -\frac{1}{\cancel{3}}(-\cancel{3}) \Rightarrow 1$$

Putting values of A & B in equ ①.

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}} \rightarrow ③$$

=> Before determining the inverse, we must specify the ROC associated with each of the term.

=> As ROC of $X(z)$ lies outside the outer most pole, the ROC for each individual term in equ ③ must lie outside the pole associated with that term.

-> Now, $\quad x[n] = x_1[n] + x_2[n]$

-> Where, $\quad x_1[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{4}z^{-1}} \quad , \quad |z| > \frac{1}{4}$

$$x_2[n] \overset{z}{\longleftrightarrow} \frac{2}{1-\frac{1}{3}z^{-1}} \quad , \quad |z| > \frac{1}{3}$$

-> By inspection, ~~we~~ ~~know~~

$$\therefore a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n] \quad , \quad x_2[n] = 2\left(\frac{1}{3}\right)^n u[n]$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

EXAMPLE #2:-

Now consider the same $X(z)$ as mentioned in Example 1 but with ROC $|z| < 1/4$.

SOL:-

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \quad , \quad |z| < \frac{1}{4}$$

→ In this case the ROC is inside both poles. i.e the points in the ROC all have magnitude smaller than either of the poles at $z = 1/3$ or $z = 1/4$.

→ Consequently, the ROC for each term in the partial fraction expansion of example 1 must also be inside the corresponding pole.

→ Now, $x_1[n]$ is given by,

$$x_1[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| < \frac{1}{4}$$

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1], \; x_2[n] = -2\left(\frac{1}{3}\right)^n u[-n-1]$$

→ then,

$$x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

EXAMPLE #3:-

$$X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty$$

SOL:-

→ From the power series definition of the z-transform, we can determine the inverse transform of $X(z)$ by inspection,

$$x[n] = \begin{cases} 4 & , n = -2 \\ 2 & , n = 0 \\ 3 & , n = 1 \\ 0 & , \text{otherwise} \end{cases}$$

that is,

$$x[n] = 4\,\delta[n+2] + 2\,\delta[n] + 3\,\delta[n-1]$$

→ z-transform pair $\delta[n+n_0] \overset{z}{\longleftrightarrow} z^{n_0}$ can also be used to simplify the above inverse transform.

EXAMPLE #4:-

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

Expand in power series by long division.

Sol:-

$$X(z) = \frac{1}{1-az^{-1}}$$

$az^{-1}(1-az^{-1})$

$az^{-1} - a^2z^{-2}$

→ By using long division,

$$1-az^{-1} \overline{)\,1\,} = 1 + az^{-1} + a^2z^{-2} + \cdots\cdots$$

$1 - az^{-1}$

$+az^{-1}$

$az^{-1} - a^2z^{-2}$

a^2z^{-2}

$a^2z^{-2} - a^3z^{-3}$

a^3z^{-3}

$\vdots$

or,

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots\cdots$$

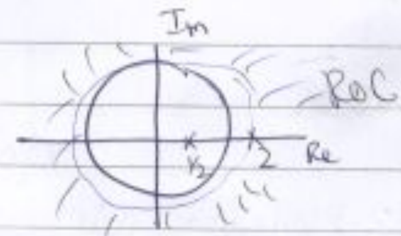
⇒ The series expansion converges since $|z| > |a|$ or equivalently $|az^{-1}| < 1$

⇒ By matching terms in power of z, we see that $x[n] = 0$ $n < 0$, $x[0] = 1$, $x[1] = a$, $x[2] = a^2$ and in general $x[n] = a^n u[n]$.

EXAMPLE # 5:-

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$h[n] = ?$



Sol:-

⇒ Since the ROC for this system function is the exterior of a circle outside the outermost pole, we know that the impulse response is right sided.

⇒ To determine if the system is causal, we then need only check the other condition required for causality, namely that H(z), when expressed as a ratio of polynomials in z, has numerator degree no larger than the denominator.

→ For this example:

$$H(z) = \frac{1-2z^{-1}+1-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

$$H(z) \Rightarrow \frac{2z^2-5/2\, z}{z^2-\frac{5}{2}z-1}$$

→ The numerator and denominator of $H(z)$ are both of degree two, and consequently we can conclude that the system is causal.

→ this can also be verified by calculating the inverse transform of $H(z)$.

→ Using pair 5 in table of standard z-transform pairs, the impu response of this system is,

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[n].$$

→ Since $h[n]=0$ for $n<0$, we can confirm that the system is causal.

Stability:-

→ If we consider the example #5 for checking stability of the system we see that since the ROC associated is the region $|z|>2$ which does not include the unit circle, the system is not stable.

EXAMPLE #6:-

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

Sol:-

→ Applying the z-transform to both sides and using the linearity property and time shifting property we see that:-

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1-\frac{1}{2}z^{-1}\right] = X(z)\left[1+\frac{1}{3}z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

→ This is the algebraic expression for $H(z)$, but not the ROC.

→ Also there are two distinct impulse responses, one right sided and the other is left sided.

→ So, there are two different choices for ROC, one $|z| \geq 1/2$ associated with assumption that $h[n]$ is right sided and the other $|z| < 1/2$ associated with the assumption that $h[n]$ is left sided.

→ Consider, the first choice of ROC equal to $|z| > 1$, writing:-

$$H(z) = \left(1+\frac{1}{3}z^{-1}\right)\frac{1}{1-\frac{1}{2}z^{-1}}$$

→ We can use transform pair 5, together with the linearity and time shifting properties, we get

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

→ For the other choice of ROC, namely $|z| < 1$, we can use transform pair 6, we get

$$h[n] = -\left(\frac{1}{2}\right)u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

→ In this case the system is anti-causal - $h[n] = 0$ for $n > 0$ and unstable.

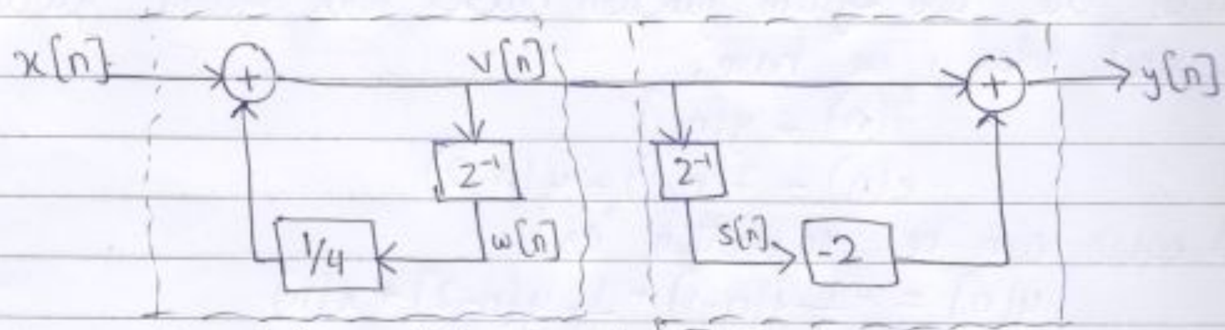
EXAMPLE # 7:-

$$H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)(1-2z^{-1})$$

SOL:-

→ The above system is the cascade of a system with system function $\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$ and one with system function $(1-2z^{-1})$.

→ Block diagram representation of above system is:



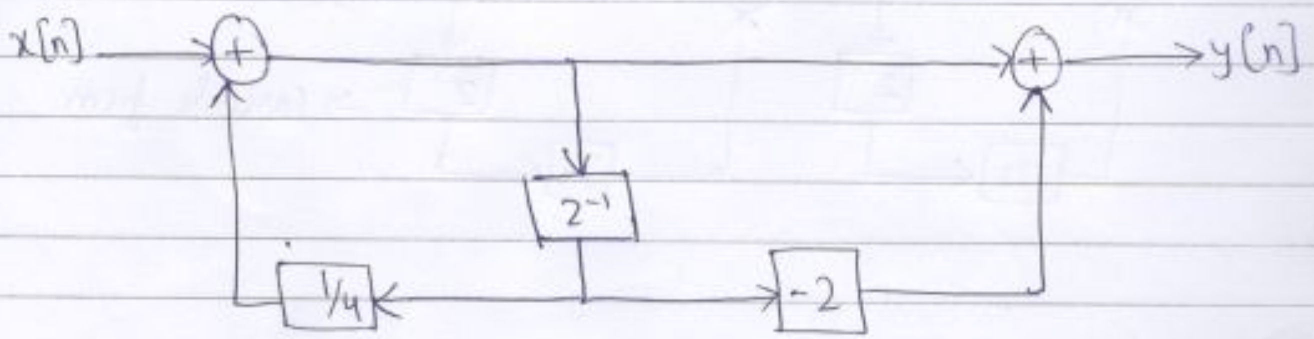
$$y[n] = v[n] - 2v[n-1]$$

→ As the input to both unit delay elements is $v[n]$, so that the outputs of these elements are identical i-e

$$w[n] = s[n] = v[n-1]$$

→ So we don't need both these delays elements and we can simply use the output of one of them as the signal to be fed to both coefficient multipliers.

→ Equivalent block diagram representation using only one unit delay element



EXAMPLE #8:-

$$H(z) = \frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

SOL:-

$\Rightarrow$ The associated difference equation of $H(z)$ is :-

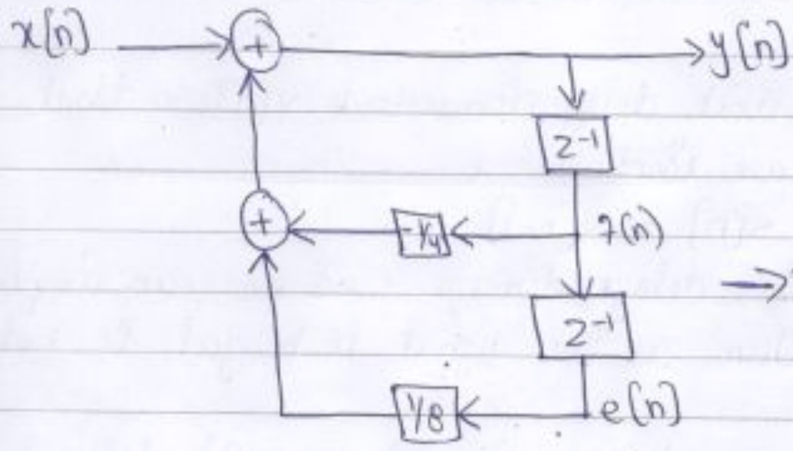
$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] \rightarrow (1)$$

$\Rightarrow$ Since the two system function blocks with system function z^{-1} are unit delay, we have,

$$7(n) = y[n-1]$$
$$e(n) = 7(n-1) = y(n-2)$$

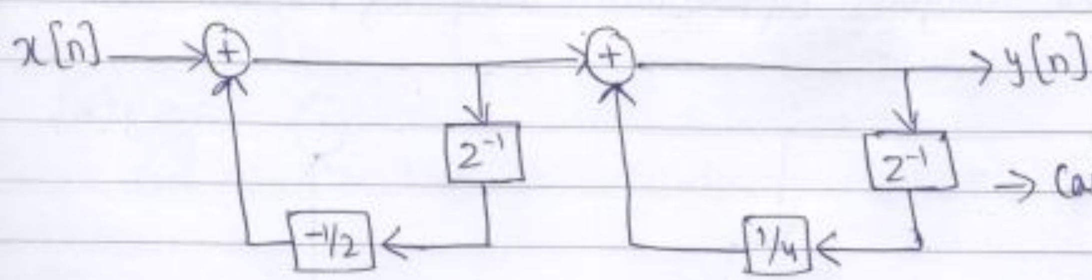
$\Rightarrow$ And equ(1) can be re written as:

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n]$$



Block diagram representation for the system H(z).

$\Rightarrow$ This form is also known as Direct form.



$\Rightarrow$ Cascade form.

→ after performing partial fraction we get, (for parallel form)

$$H(z) = \frac{2/3}{1+\frac{1}{2}z^{-1}} + \frac{1/3}{1-\frac{1}{4}z^{-1}}$$