ISRA UNIVERSITY
Islamabad Campus

Department of Electrical Engineering
Program: B.E. (Electrical)
Semester - Fall 2016

EL313- Signal & Systems

Assignment – 3 Solution | **Due Date: 02/01/2017**
Marks: 20 | **Handout Date: 26/12/2016**

Question # 1:

Using partial fraction expansion and the fact that:

$$(a)^n u[n] \leftrightarrow \frac{1}{1-az^{-1}} \quad , \ |z| > |a|$$

Find the inverse z-transform of:

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} \ , \ |z| > 2$$

Solution:

Using partial fraction expansion:

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} = \frac{A}{(1-z^{-1})} + \frac{B}{(1+2z^{-1})} \rightarrow eq(1)$$

Cross multiplication gives:

$$1-\frac{1}{3}z^{-1} = A(1+2z^{-1}) + B(1-z^{-1})$$

Putting $z^{-1} = 1$, gives:

$$1-\frac{1}{3}(1) = A\big(1+2(1)\big) + B(1-1)$$

$$\frac{3-1}{3} = A(1+2) + B(0)$$

$$\frac{2}{3} = A(3) \Rightarrow A = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

Putting $z^{-1} = -\frac{1}{2}$, gives:

$$1-\frac{1}{3}\left(-\frac{1}{2}\right) = A\left(1+2\left(-\frac{1}{2}\right)\right) + B\left(1-\left(-\frac{1}{2}\right)\right)$$

$$1+\frac{1}{6} = A(0) + B\left(1+\frac{1}{2}\right)$$

$$\frac{6+1}{6} = B\left(\frac{2+1}{2}\right)$$

$$\frac{7}{6} = B\left(\frac{3}{2}\right) \Longrightarrow A = \frac{7}{6} \times \frac{2}{3} = \frac{7}{9}$$

Putting values of A and B in eq(1) gives:

$$X(z) = \frac{\frac{2}{9}}{(1 - z^{-1})} + \frac{\frac{7}{9}}{(1 + 2z^{-1})}$$

Taking the inverse z-transform, we obtain:

$$x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n]$$

Question # 2:

Determine the z-transform for each of the following sequences. Sketch the pole-zero plot and indicate the ROC:

i. $\delta[n-5]$
ii. $(-1)^n u[n]$
iii. $\left(\frac{1}{2}\right)^n u[n]$

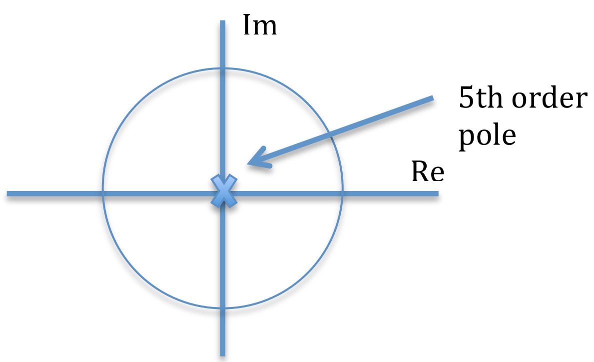
Solution:

i. $\delta[n-5]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

$$X(z) = \sum_{n=5}^{\infty} \delta[n-5]z^{-1} \Longrightarrow z^{-5}$$

$$Region\ of\ Covergence\ is\ all\ z\ except\ 0.$$



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ii. $(-1)^n u[n]$

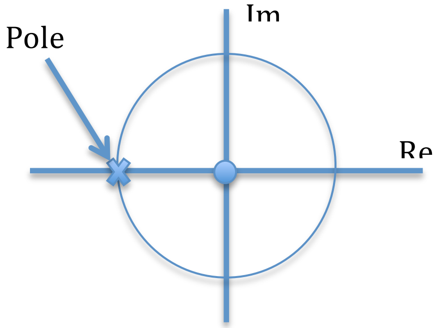
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} (-1)^n u[n]z^{-1}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^{-1} \Longrightarrow \frac{1}{1+z^{-1}}$$

$$zeros\ is\ at\ 0\ and\ pole\ is\ at\ 1+z^{-1} = 0 \Longrightarrow z = -1$$

$$Region\ of\ Covergence = |z| > 1$$



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iii. $\left(\frac{1}{2}\right)^n u[n]$

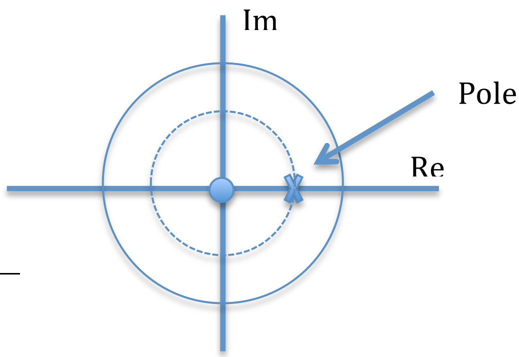
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]z^{-1}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-1} \Longrightarrow \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$zeros\ is\ at\ 0\ and\ pole\ is\ at\ 1-\frac{1}{2}z^{-1} = 0 \Longrightarrow z = \frac{1}{2}$$

$$Region\ of\ Covergence = |z| > \frac{1}{2}$$



Question # 3:

Using the partial fraction expansion, determine the sequence x [n] that goes with the following z-transform:

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}$$

Solution:

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}$$

Multiply and dividing given transform with z^{-1} gives:

$$X(z) = \frac{3z^{-1}}{z(z^{-1}) - \frac{1}{4}(z^{-1}) - \frac{1}{8}z^{-1}(z^{-1})} = \frac{3z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \rightarrow eq(1)$$

Factoring the denominator of eq (1) gives:

$$X(z) = \frac{3z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}$$

Since x [n] is absolutely summable, the ROC must be $|z| > \frac{1}{2}$ in order to include the unit circle.

Using partial fraction expansion:

$$X(z) = \frac{3z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)} = \frac{A}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{B}{\left(1 + \frac{1}{4}z^{-1}\right)} \rightarrow eq(2)$$

Cross multiplication gives:

$$3z^{-1} = A\left(1 + \frac{1}{4}z^{-1}\right) + B\left(1 - \frac{1}{2}z^{-1}\right)$$

Putting $z^{-1} = -4$, gives:

$$3(-4) = A\left(1 + \frac{1}{4}(-4)\right) + B\left(1 - \frac{1}{2}(-4)\right)$$

$$-12 = A(0) + B(1 + 2)$$

$$-12 = B(3) \Rightarrow B = -\frac{12}{3} = -4$$

Putting $z^{-1} = 2$, gives:

$$3(2) = A\left(1 + \frac{1}{4}(2)\right) + B\left(1 - \frac{1}{2}(2)\right)$$

$$6 = A\left(1 + \frac{1}{2}\right) + B(0)$$

$$6 = A\left(\frac{2 + 1}{2}\right)$$

$$6 = A\left(\frac{3}{2}\right) \Rightarrow A = (6)\left(\frac{2}{3}\right) = 4$$

Putting values of A and B in eq (2) gives:

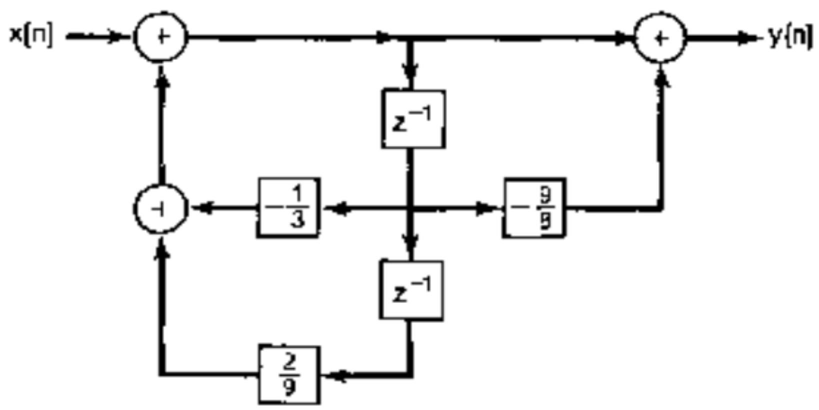
$$X(z) = \frac{4}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{-4}{\left(1 + \frac{1}{4}z^{-1}\right)} \Rightarrow \frac{4}{\left(1 - \frac{1}{2}z^{-1}\right)} - \frac{4}{\left(1 + \frac{1}{4}z^{-1}\right)}$$

Taking the inverse z-transform, we obtain:

$$x[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{4}\right)^n u[n]$$

Question # 4:

The input x [n] and output y [n] of a causal LTI system are related through the block diagram representation shown below:



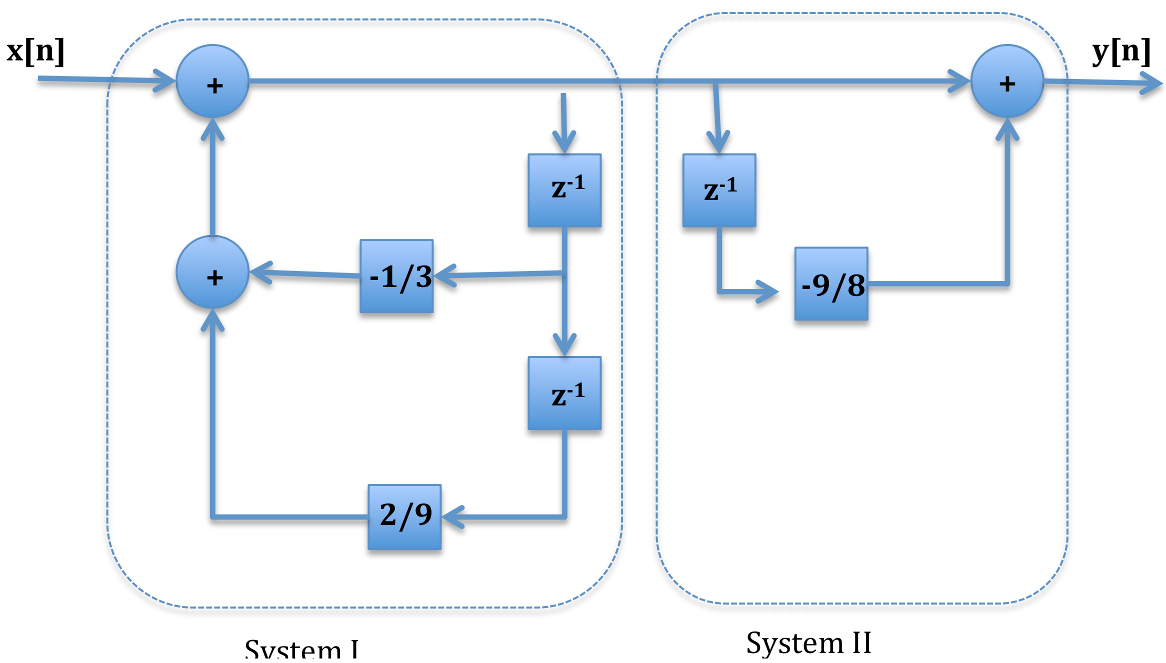
a) Determine a difference equation relating y [n] and x [n].

b) Is this system stable?

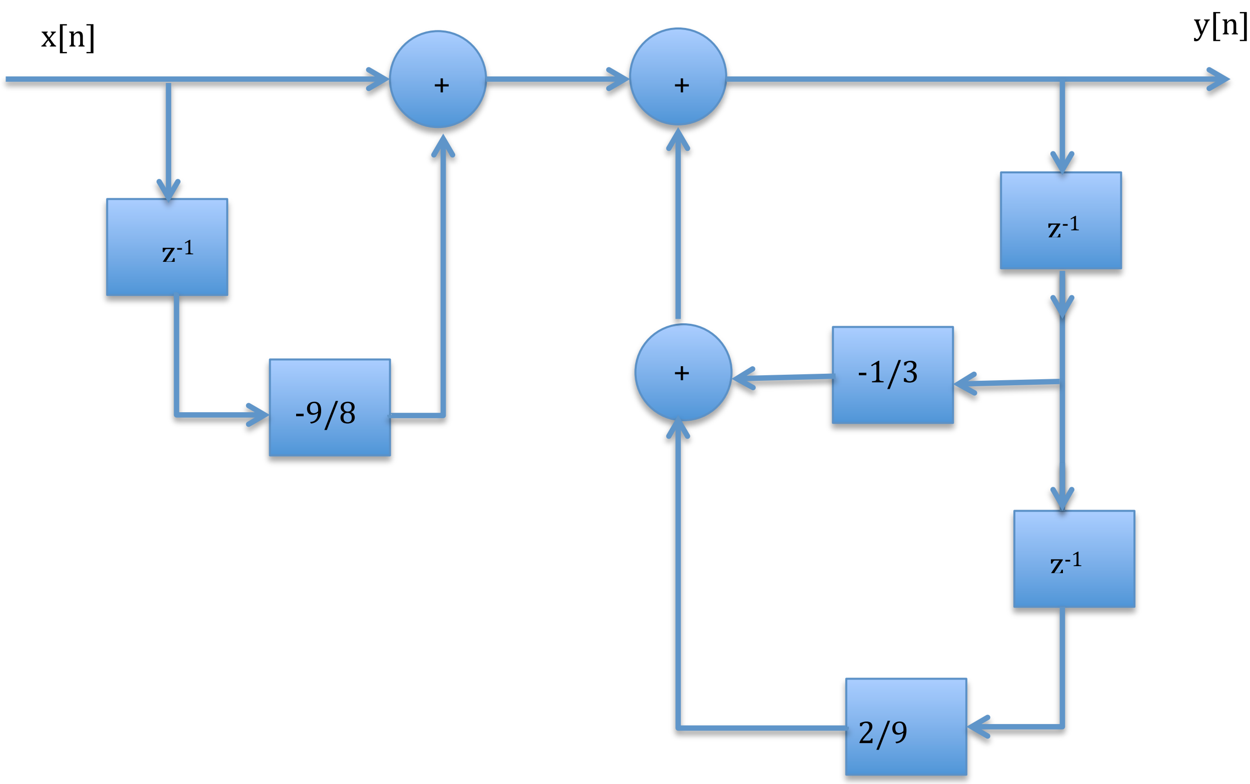
Solution:

a) Determine a difference equation relating y [n] and x [n].

The block diagram may be redrawn as shown below:



This may be treated as a cascade of the two systems shown within the dotted lines. These two systems may be interchanged as shown below without changing the system function of the overall system.



$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{9}{8}z^{-1}}{1 + \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2}}$$

$$Y(z)\left(1 + \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2}\right) = X(z)\left(1 - \frac{9}{8}z^{-1}\right)$$

$$Y(z) + \frac{1}{3}Y(z)z^{-1} - \frac{2}{9}Y(z)z^{-2} = X(z) - \frac{9}{8}X(z)z^{-1}$$

Taking inverse z-transform we get:

$$y[n] + \frac{1}{3}y[n-1] - \frac{2}{9}y[n-2] = x[n] - \frac{9}{8}x[n-1]$$

Hence the difference equation related to the system shown above is:

$$y[n] = x[n] - \frac{9}{8}x[n-1] - \frac{1}{3}y[n-1] + \frac{2}{9}y[n-2]$$

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b) Is this system stable?

For this part consider the transfer function equation, i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{9}{8}z^{-1}}{1 + \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2}} = \frac{1 - \frac{9}{8}z^{-1}}{\left(1 + \frac{2}{3}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

H (z) has poles at $z = \frac{1}{3}$ and $z = -\frac{2}{3}$.

Since the system is causal, the ROC has to be at $|z| > \frac{2}{3}$. The ROC includes the unit circle and hence the system is stable.

Good Luck