Signal & Systems
Lecture #2

13th March 18

Continuous & Discrete Signals

Classification of Signals

Periodic vs Aperiodic

- Definition-1: A continuous time signal x(t) is periodic if there is a constant T > 0 such that:

$$x(t) = x(t+T), \quad for \quad all \quad t \in R$$

- Definition-2: A discrete time signal x[n] is periodic if there is an integer constant N > 0 such that:

$$x[n] = x[n+N], \quad for \quad all \quad n \in Z$$

- Signals do not satisfy the periodicity conditions are called aperiodic signals.

- T_0 is called the fundamental period of x(t) if it is the smallest value of T >0 satisfying the periodicity condition. The number $\omega_0 = \frac{2\pi}{T_0}$ is called the fundamental frequency of x(t).

Periodic vs Aperiodic (cont.)

- N_o is called the fundamental period of x[n] if it is smallest value of N > o where N ε Z satisfying the periodicity condition. The number $\frac{\Omega_0}{2\pi} = \frac{m}{N}$ is called the fundamental frequency of x[n].

Example #1

- Determine the fundamental period of the following signals:

$$(a): \quad e^{j3\pi t/5}$$

$$(b): \quad e^{j3\pi n/5}$$

Even & Odd Signals

- An even signal is any signal f such that f(t) = f(-t).
- A signal x(t) or x[n] is referred to as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin.
- An odd signal on the other hand is a signal f such that f(t) = -(f(-t)).
- Any signal can be written as a combination of an even and odd signal, i.e., every signal has an odd-even decomposition.

$$f\left(t\right)=\frac{1}{2}\left(f(t)+f(-t)\right)+\frac{1}{2}(f(t)-f(-t))$$

Even & Odd Signals (cont.)

- The all-zero signal is both even and odd. Any other signal cannot be both even and odd, but may be neither.

Energy & Power

- The total energy of a continuous time signal x(t) , where x(t) is defined for $-\infty < t < \infty$, is

$$E_\infty = \int_{-\infty}^{\infty} x^2(t)\,dt = \lim_{T\to\infty} \int_{-T}^{T} x^2(t)\,dt$$

- The time-average power of a signal is:

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

- An energy signal is a signal with finite E_∞. For an energy signal, P_∞ =0.

- A power signal is a signal with finite, nonzero P_∞. For a power signal, $E_\infty=\infty$.

Energy & Power (cont.)

- The total energy of a discrete-time signal is defined by:

$$E_\infty = \sum_{n=-\infty}^{\infty} x^2[n] = \lim_{N\to\infty} \sum_{n=-N}^{N} x^2[n]$$

- The time-average power is:

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

Continuous-Time Complex Exponential

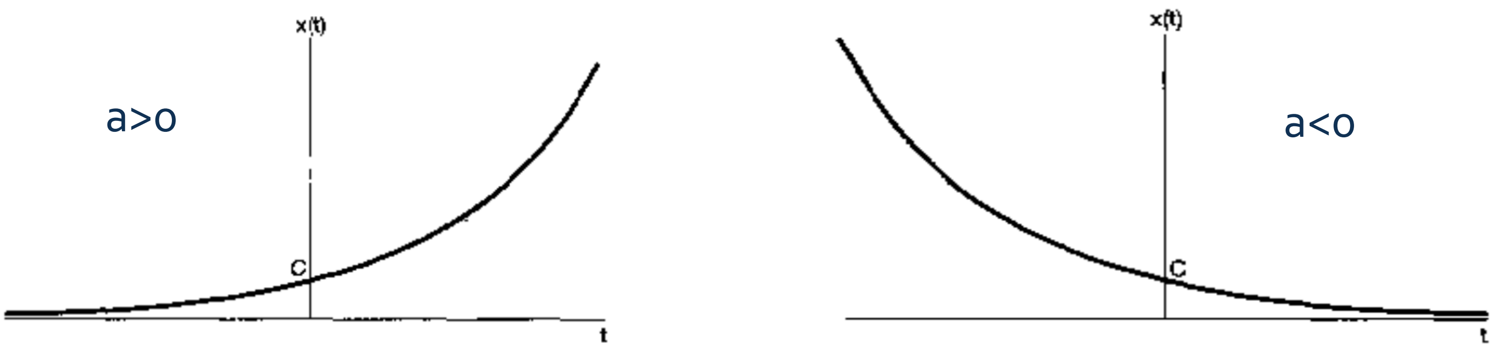
- The continuous-time complex exponential signal is of the form:

$$x(t) = Ce^{at}, \quad where \quad C, \quad a \in C$$

- Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

Real Exponential Signals

- If C and a are real there are basically two types of behaviour.
- If a is positive, then as t increase x(t) is a growing exponential, i.e., when a>o.
- If a is negative then x(t) is a decaying exponential, i.e., when a<o.
- When a=o then x(t) is constant.



Periodic Complex Exponential

- Let's consider the case where a is purely imaginary, i.e., $a = j\omega_o$, ω_o belongs to R.
- Since C is a complex number, we have: $C = Ae^{j\theta}$ where A, θ belongs to R.
- Consequently: $x(t) = Ce^{j\omega_0 t} = Ae^{j\theta}e^{j\omega_0 t}$

$$= Ae^{j(\omega_0 t+\theta)} = A\cos(\omega_0 t+\theta) + jA\sin(\omega_0 t+\theta)$$

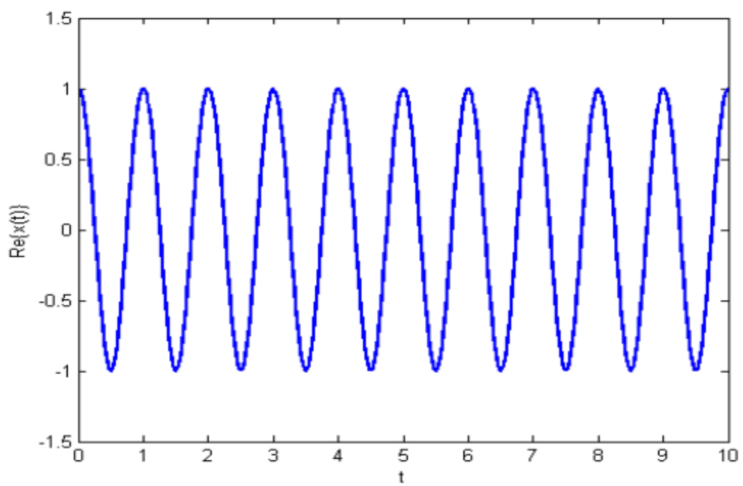
- The real and imaginary parts of x(t) are:

$$\mathrm{Re}\{x(t)\} = A\cos(\omega_0 t+\theta)$$

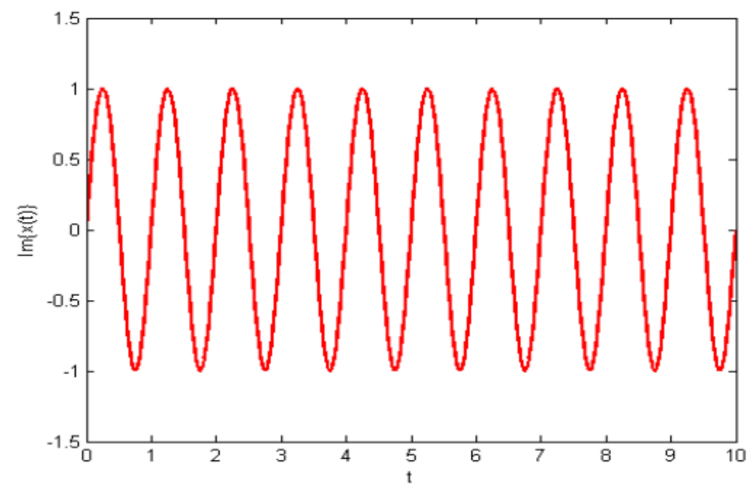
$$\mathrm{Im}\{x(t)\} = A\sin(\omega_0 t+\theta)$$

Periodic Complex Exponential (cont.)

- We can think of x(t) as a pair of sinusoidal signals of the same amplitude A, ω_o and phase shift θ with one a cosine and the other a sine.



(a) $\mathcal{R}e\{Ce^{j\omega_0 t}\}$



(b) $\mathcal{I}m\{Ce^{j\omega_0 t}\}$

Periodic complex exponential function $x(t)= Ce^{j\omega_o t}$, $C=1$, $\omega_o=2\pi$

Periodic Complex Exponential (cont.)

- $x(t) = Ce^{j\omega_0 t}$ is periodic with:
  - Fundamental period: $T_o = 2\pi/|\omega_o|$
  - Fundamental frequency: $|\omega_o|$

- the second claim is the immediate result from the first claim. To show the first claim, we need to show that $x(t+T_o) = x(t)$ and no smaller T_o can satisfy the periodicity criteria.

$$x(t+T_0) = Ce^{j\omega_0\left(t+\frac{2\pi}{|\omega_0|}\right)} = Ce^{j\omega_0 t}e^{\pm j2\pi}$$

$$= Ce^{j\omega_0 t} = x(t)$$

- It is easy to show that T_o is the smallest period.

General Complex Exponential

- The most general case of a complex exponential can be expressed and interpreted in terms of the two cases: the real exponential and the periodic complex exponential.
- Consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. I.e., $C = |C|e^{j\theta}$
- And:

$$a = r + j\omega_0$$

- Then:

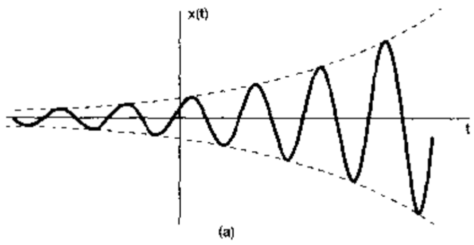
$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

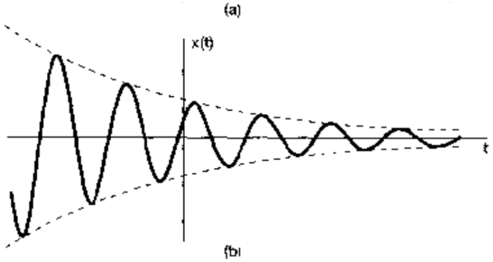
- Using Euler's relation, we can expand this further as:

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

General Complex Exponential(cont.)

- Thus for r=o, the real and imaginary parts of a complex exponential are sinusoidal.
- For r>o they correspond to sinusoidal signals multiplied by a growing exponential.
- For r < o, they correspond to sinusoidal signals multiplied by a decaying exponential.
- As shown below: (a) is growing sinusoidal signal when r>o, (b) is decaying sinusoid when r<o.





General Complex Exponential(cont.)

- Sinusoidal signals multiplied by decaying exponentials are commonly referred to as damped signals.

Discrete-Time Complex Exponential

- A discrete-time complex exponential function has the form:

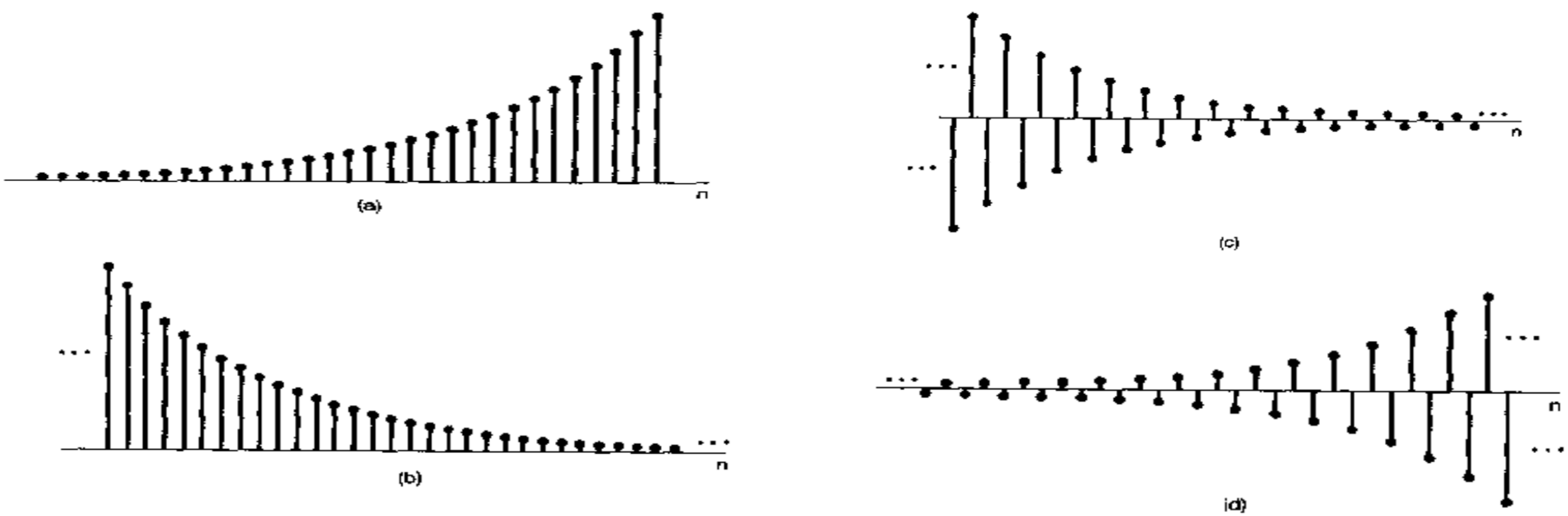
$$x[n] = Ce^{\beta n}$$

- Where C, β belongs to Complex. Letting $\alpha = e^{\beta}$:

$$x[n] = C\alpha^{n}$$

Real-Valued Complex Exponential

- x[n] is a real-valued complex exponential when C belongs to R and α belongs to R.
- In this case, $x[n]=C\alpha^n$ is a monotonic decreasing function when $0 < \alpha < 1$ and is a monotonic increasing when $\alpha > 1$.



The real exponential signal (a) $\alpha>1$, (b) $0< \alpha <1$, (c) $-1< \alpha <0$, (d) $\alpha < -1$

Complex-Valued Complex Exponential

- x[n] is a complex-valued complex exponential when C,α belongs to complex.
- In this case C and α can be written as:

$$C = |C| e^{j\theta} \quad and \quad \alpha = |\alpha| e^{j\Omega_0}$$

Comsequently,

$$x[n] = C\alpha^n = |C| e^{j\theta} \left(|\alpha| e^{j\Omega_0} \right)^n$$

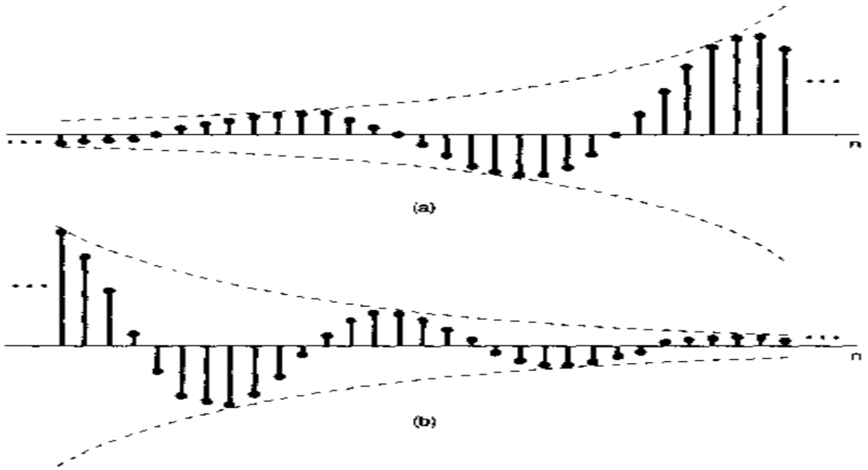
$$= |C||\alpha|^n e^{j(\Omega_0 n + \theta)}$$

$$= |C||\alpha|^n \cos(\Omega_0 n + \theta) + j|C||\alpha|^n \sin(\Omega_0 n + \theta)$$

Complex-Valued Complex Exponential(cont.)

- Three cases can be considered here:
  - When $|\alpha|=1$, then $x[n] = |C|\cos(\Omega_0 n+\theta) + j\,|C|\sin(\Omega_0 n+\theta)$ and it has sinusoidal real and imaginary parts (not necessarily periodic though).
  - When $|\alpha| > 1$, then $|\alpha|^n$ is a growing exponential, so the real and imaginary parts of x[n] are the product of this with sinusoids.
  - When $|\alpha| < 1$, then the real and imaginary parts of x[n] are sinusoids sealed by a decaying exponential.

(a) Growing Discrete-time sinusoidal signals (b) decaying discrete time sinusoid



Periodic Complex Exponential

- Consider $x[n] = Ce^{j\Omega_0 n}, \Omega_0 \in R$ We want to study the condition for x[n] to be periodic.

- The periodicity condition requires that, for some N>o,

$$x[n+N] = x[n], \quad \forall n \in Z$$

- Since $x[n] = Ce^{j\Omega_0 n}$, it holds that:

$$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} e^{j\Omega_0 N} = e^{j\Omega_0 n}, \quad \forall n \in Z$$

- This is equivalent to:

$$e^{j\Omega_0 N} = 1 \quad or \quad \Omega_0 N = 2\pi m, \quad for \quad some \quad m \in Z$$

Periodic Complex Exponential (cont.)

- Therefore, the condition for periodicity of x[n] is:

$$\Omega_0 = \frac{2\pi m}{N}$$

  - For some m belongs to Z and some N>o, N belongs to Z.

Periodic Complex Exponential (cont.)

- Thus $x[n] = e^{j\Omega_o n}$ is periodic if and only if Ω_o is a rational multiple of 2π.
- The fundamental period is:

$$N = \frac{2\pi m}{\Omega_0}$$

- Where we assume that m and N are relatively prime, gcd (m,n) =1, i.e., m/N is in reduced form.

Impulse & Step Functions

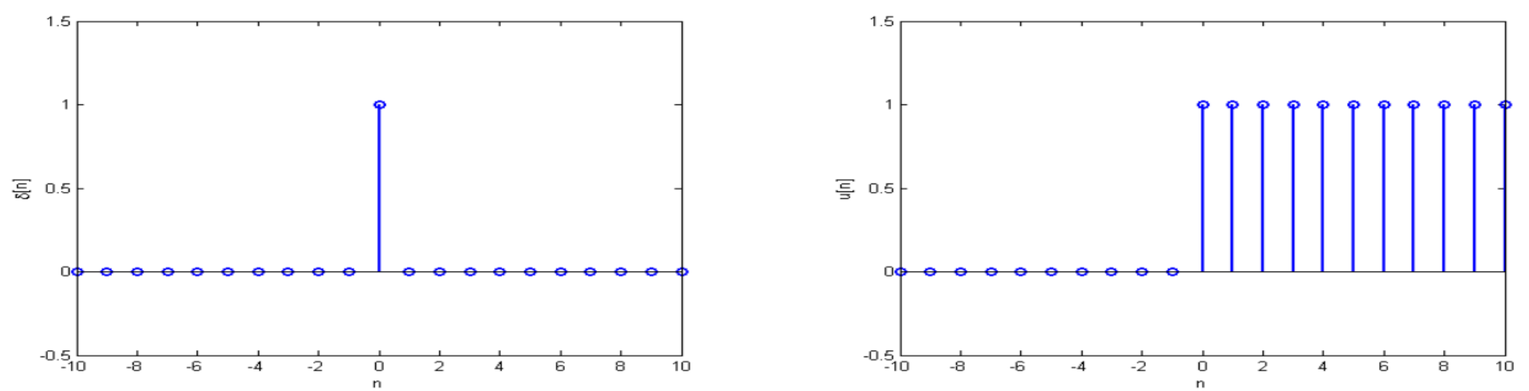
Discrete-Time Impulse & Step Functions

- The discrete-time unit impulse signal δ[n] is defined as:

$$\delta(n) = \begin{cases} 1 & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases}$$

- The discrete-time unit step signal u[n] is defined as:

$$u(n) = \begin{cases} 1 & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$



(a) $\delta[n]$ (b) $u[n]$

Relation B/w Unit Impulse & Unit Step Sequences

- Discrete time unit impulse is the first difference of the discrete time unit step. I.e.; $\delta[n]=u[n]-u[n-1]$

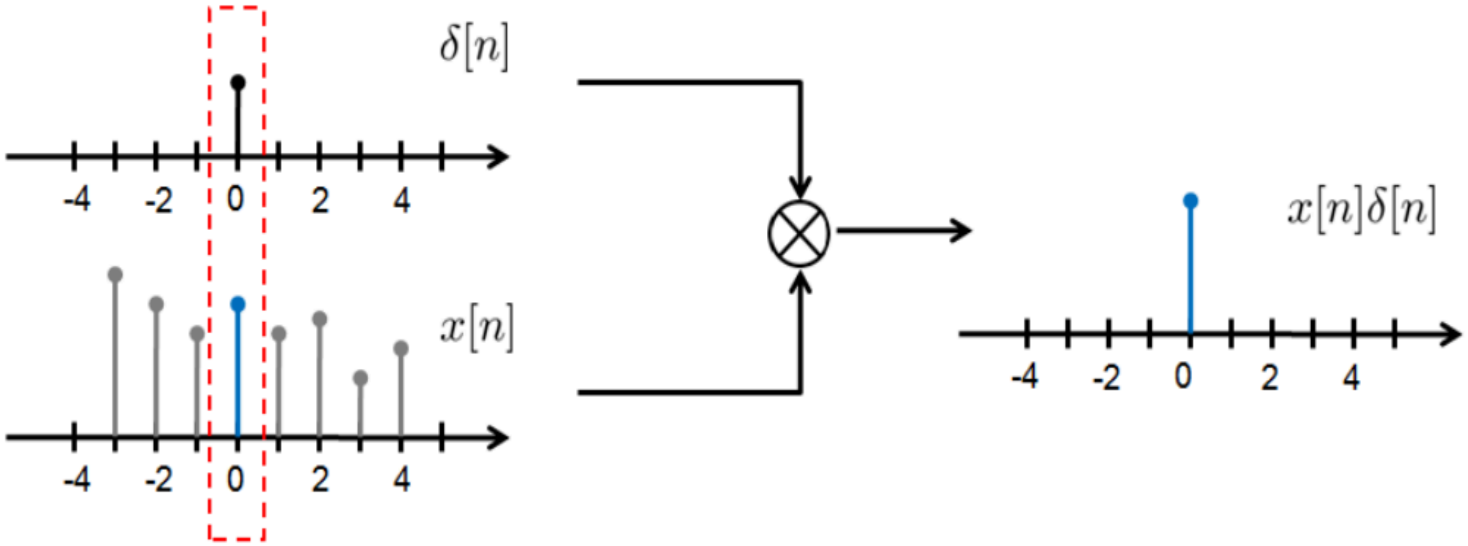
- Discrete time unit step is the running sum of the discrete time unit impulse or unit sample. i.e.;

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Property of δ[n]

- **Sampling Property:**
- By the definition δ[n], $\delta[n-n_o] = 1$ if $n=n_o$ and 0 otherwise.
- Therefore, $$x[n]\delta[n-n_0] = \begin{cases} x[n], & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$
$$= x[n_0]\delta[n-n_0]$$
- As a special case when $n_o=0$, we have x[n] δ[n]=x[0] δ[n].
- When a signal x[n] is multiplied with δ[n], the output is a unit impulse with amplitude x[0].

Property of δ[n] (cont.)



Property of δ[n] (cont.)

- **Shifting Property:**
- Since x[n] δ[n]= x[o] δ[n] and $\sum_{n=-\infty}^{\infty} \delta[n] = 1$, we have

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n] = \sum_{n=-\infty}^{\infty} x[0]\delta[n] = x[0]\sum_{n=-\infty}^{\infty} \delta[n] = x[0]$$

- And similarly:

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = \sum_{n=-\infty}^{\infty} x[n_0]\delta[n-n_0] = x[n_0]$$

- In general, the following result holds:

$$\sum_{n=a}^{b} x[n]\delta[n-n_0] = \begin{cases} x[n_0], & if \quad n_0 \in [a,b] \\ 0, & if \quad n_0 \notin [a,b] \end{cases}$$

Continuous-Time Impulse & Step Functions

- The Dirac delta is defined as:

$$\delta(t) = \begin{cases} 1 & for \quad t = 0 \\ 0 & for \quad t \neq 0 \end{cases}$$

- Where:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- The unit step function is defined as:

$$u(t) = \begin{cases} 1 & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

Property of δ(t)

- The properties of δ(t) are analogous to the discrete-time case:
- **Sampling Property:**

$$x(t)\delta(t) = x(0)\delta(t)$$

- Note that x(t) δ(t) = x(o) when t=o and x(t) δ(t) =o when t≠o.
- Similarly we have:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$for \quad any \quad t_0 \in R$$

Property of δ(t) (cont.)

- **Shifting Property:**
  - The shifting property follows from the sampling property.
  - Integrating x(t) δ(t) yields:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = \int_{-\infty}^{\infty} x(0)\delta(t)\,dt = x(0)\int_{-\infty}^{\infty} \delta(t)\,dt = x(0)$$

  - Similarly, one can show that:

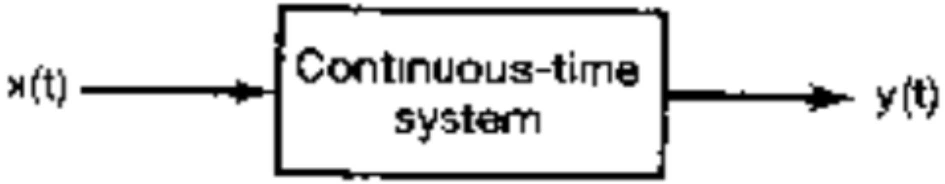
$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

Continuous & Discrete Systems

Fundamentals of Systems

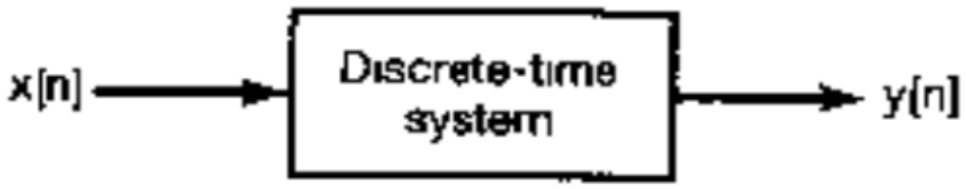
Systems

- A system in the broadcast sense are an interconnection of components, devices or subsystems.
- A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way resulting in other signals as output.
- A continuous time system is a system in which continuous time input signals are applied and result in continuous time output signals. The input-output relation is represented by the following notation: $x(t) \rightarrow y(t)$.



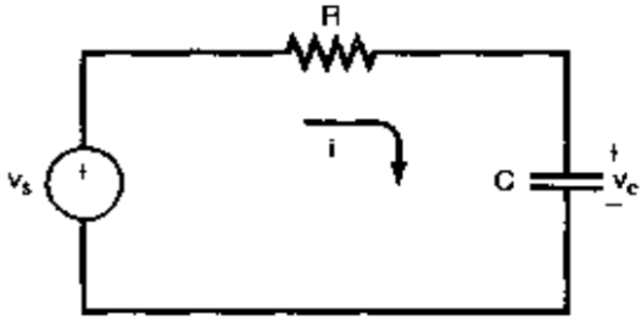
Systems (cont.)

- Similarly a discrete time system is a system that transforms discrete time inputs into discrete time outputs and represented symbolically as: x[n]➔y[n].



Example #2

- Consider the RC circuit depicted below:



- If we regard $v_s(t)$ as the input signal and $v_c(t)$ as the output signal, then we can use simple circuit analysis to derive an equation describing the relationship between the input and output.

- Specifically from Ohm's law the current i(t) through the resistor is proportional (with proportionality constant 1/R) to the voltage drop across the resistor; i.e.,

Example #2 (cont.)

$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

- Similarly, using the defining constitutive relation for a capacitor, we can relate i(t) to the rate of change with tome of the voltage across the capacitor:

$$i(t) = C\frac{dv_c}{dt}$$

- Equating the right hand side of above two equations, we obtain a differential equation describing the relationship between the input $v_s(t)$ and the output $v_c(t)$:

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

Thank You