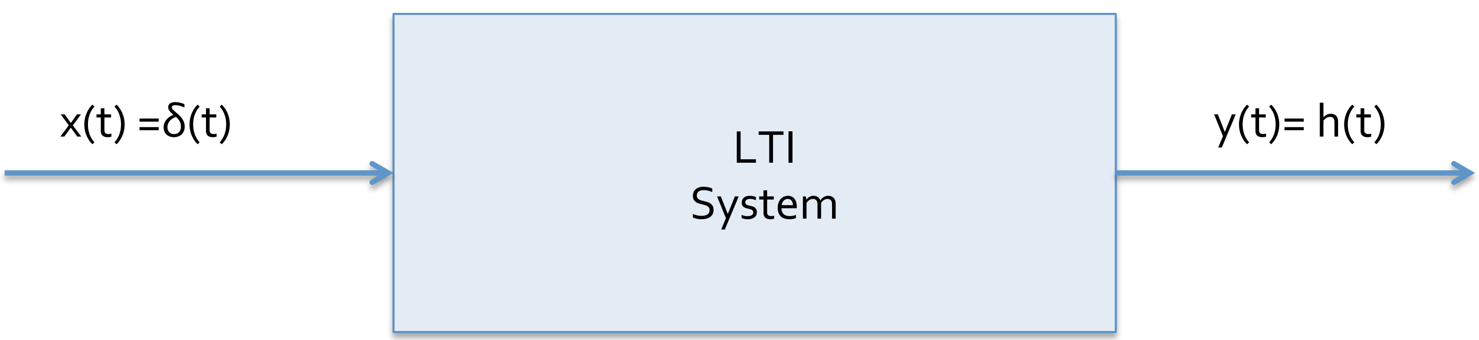
Signal & Systems
Lecture #4

27th March 18

LTI Systems

LTI Systems

- LTI= Linear Time Invariant Systems
- As Linear systems follow the principle of superposition, hence LTI will also follow the principle of superposition.
- Any delay in the input is reflected in output, property of Time invariant system will also be followed by LTI system.



Impulse Response

- Impulse response is the output of an LTI system and it is only related to LTI systems.
- Unit impulse signal is the input of LTI system and impulse response h(t) is the output or response of unit impulse signal.

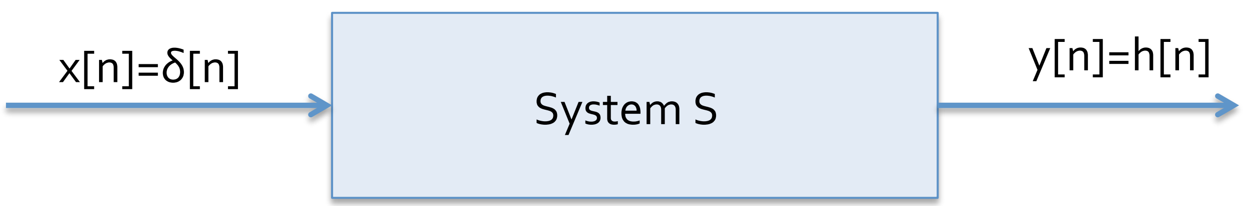
Convolution

Introduction

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.
- The input signal can be decomposed into a set of impulses, each of which is scales and shifted delta/impulse function.
- The output from each impulse is scaled and shifted version of the impulse response.
- Then the overall output signal can be added to form one output.
- That is if we know the system's impulse response we can calculate the output for any possible input.
- This response is known as convolution kernel.

The Convolution Sum

- Denote by h[n] the "impulse response" of an LTI system S.
- The impulse response is the response of the system to a unit impulse input.
- The definition of an unit impulse is: $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



- Let consider:

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

The Convolution Sum (cont.)

- Using the above favt we get the following equalities:

$$x[n]\delta[n] = x[0]\delta[n], \quad (n_0 = 0)$$

$$x[n]\delta[n-1] = x[1]\delta[n-1], \quad (n_0 = 1)$$

$$x[n]\delta[n-2] = x[2]\delta[n-2], \quad (n_0 = 2)$$

$$\vdots \qquad\qquad \vdots$$

$$= x[n]\left(\sum_{k=-\infty}^{\infty} \delta[n-k]\right) \qquad = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The sum on the left hand side is: $x[n]\left(\sum_{k=-\infty}^{\infty} \delta[n-k]\right) = x[n]$
- Because $\sum_{k=-\infty}^{\infty} \delta[n-k] = 1$ for all n.

The Convolution Sum (cont.)

- The sum on the right hand side is: $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- Therefore, equating the left hand side and right hand side yields:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- Any signal x[n] can be expressed as a sum of impulses.
- Suppose we know that the impulse response of an LTI system is h[n] and we want to determine the output y[n].
- To do so we first express x[n] as a sum of impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The Convolution Sum (cont.)

- For each impulse δ [n-k], we can determine its impulse response, because for an LTI system: $\delta[n-k] \to h[n-k]$
- Consequently, we have:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \to \sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n]$$

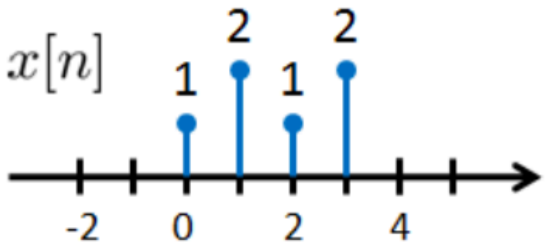
- Where the equation: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is known as the convolution sum / equation.

How to Evaluate Convolution?

- There are three basic steps used to evaluate convolution of any signal:
  - Flip
  - Shift
  - Multiply and Add.

Example #1

- Consider the signal x[n] and the impulse response h[n] shown below:



Continuous-Time Convolution

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)$$

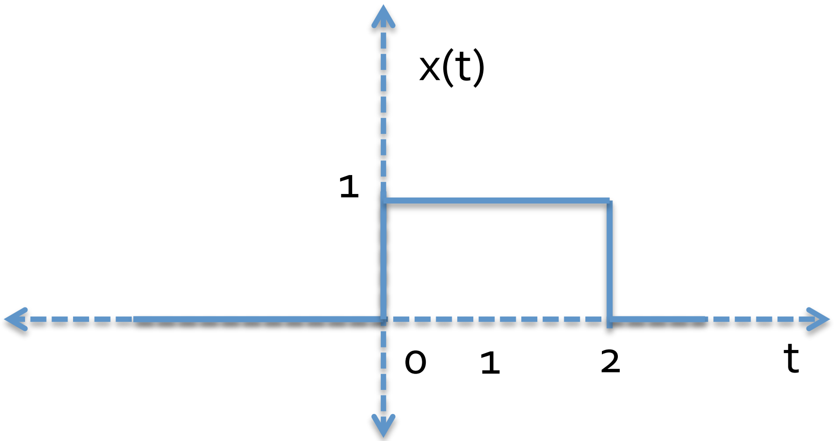
- "*" is the convolution operator.
- Replace t by dummy variable τ, as we are going to shift our signal waveform and time will vary so to avoid confusion we are replacing t.

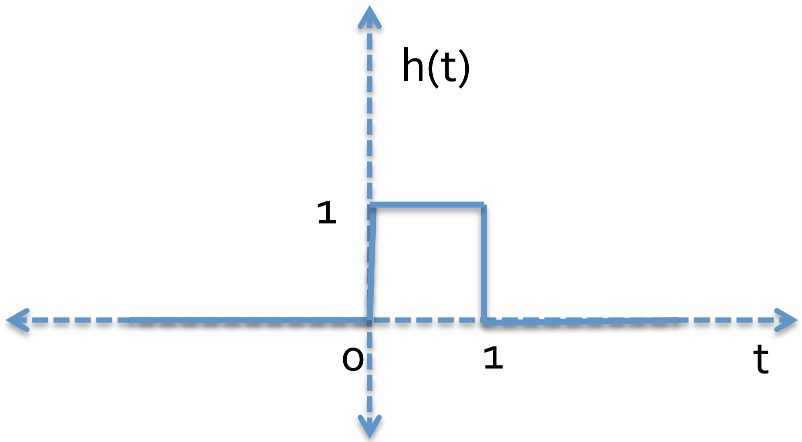
Steps of Convolution

- Step #1: Replace t by τ.
- Step #2: Perform the time reversal on one of the signal.
- Step #3: Perform the time shifting against τ on the reversed signal.
- Step #4: Multiply x(τ) and h(t-τ).
- Step #5: Integrate them.

Example #2

- x(t) ➔ i/p…. h(t) ➔ o/p (impulse response) ….y(t) =?





Properties of Convolution

Properties

- The convolution properties are as follows:
  - Commutative Property
  - Associative Property
  - Distributive Property

Commutative Property

- A basic property of convolution in both continuous and discrete time is that it is a commutative operation.
- In discrete time it is: $x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty} h[k]x[n-k]$
- In continuous time it is:

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Proof: In the discrete time case if we let r=n-k or equivalently k=n-r then:

$$x[n]*h[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n-k]=\sum_{r=-\infty}^{+\infty} x[n-r]h[r]=h[n]*x[n]$$

- With this substitution of variables, the roles of x[n] and h[n] are interchanged.

Commutative Property (cont.)

- This property states that one of the two forms for computing convolutions in discrete time and continuous time may be easier to visualize, but both forms always result in the same answer.

Distributive Property

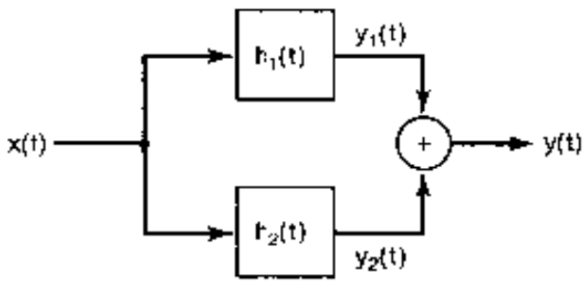
- Convolution distributes over addition so that in discrete time:

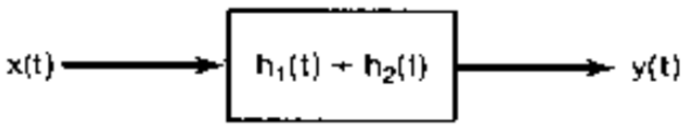
$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

- In continuous time:

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$

- Interpretation of Distributive property of convolution for a parallel interconnection of LTI systems is shown below:





Distributive Property (cont.)

- The two systems with impulse responses $h_1(t)$ and $h_2(t)$ have identical inputs and their outputs are added.
- Since:

$$y_1(t) = x(t) * h_1(t)$$

$$and$$

$$y_2(t) = x(t) * h_2(t)$$

- The system of above figure has output:

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

- The system of second figure has the output:

$$y(t) = x(t) * \left[h_1(t) + h_2(t)\right]$$

Distributive Property (cont.)

- Comparing both the above results we see that the systems in above figures are identical.
- In same way distributive property of discrete time can also be proved.

Associative Property

- Another important and useful property of convolution is associative property.
- In discrete time: $x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$
- In continuous time: $x(t)*[h_1(t)*h_2(t)] = [x(t)*h_1(t)]*h_2(t)$
- As a consequence of the associative property, the expressions:

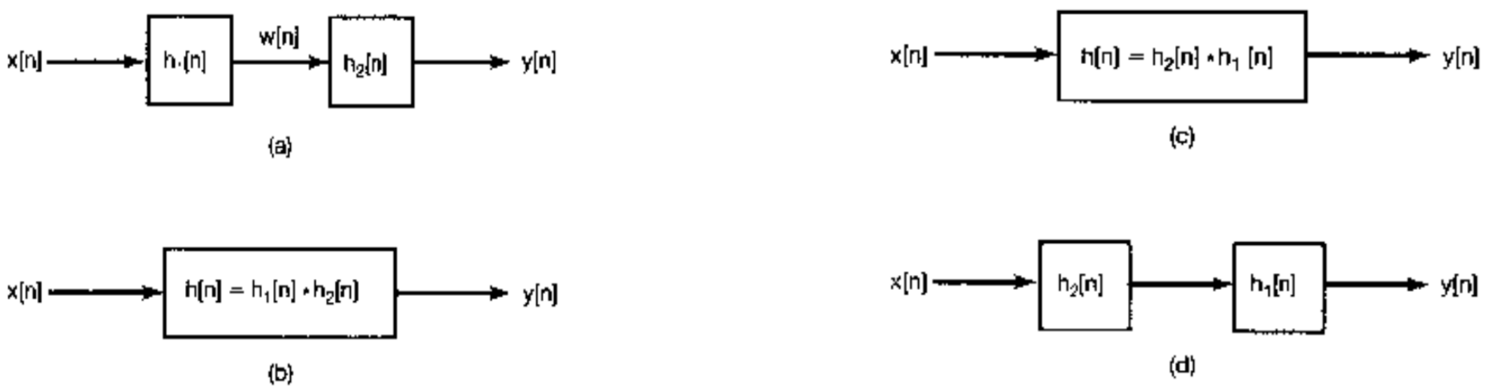
$$y[n] = x[n]*h_1[n]*h_2[n]$$

$$and$$

$$y(t) = x(t)*h_1(t)*h_2(t)$$

Associative Property (cont.)

- Are unambiguous. That is it does not matter in which order we convolve these signals.
- An interpretation of the associative property is illustrated for discrete time systems in figures below:



Associative Property (cont.)

$$y[n] = w[n] * h_2[n] = \left(x[n] * h_1[n]\right) * h_2[n], \quad fig(a)$$

$$y[n] = x[n] * h[n] = x[n] * \left(h_1[n] * h_2[n]\right), \quad fig(b)$$

- According to the associative property the series interconnection of the two systems in fig(a) is equivalent to the single system in fig(b).
- This can be generalized to an arbitrary number of LTI systems in cascade and the analogous interpretation and conclusion also hold in continuous time.

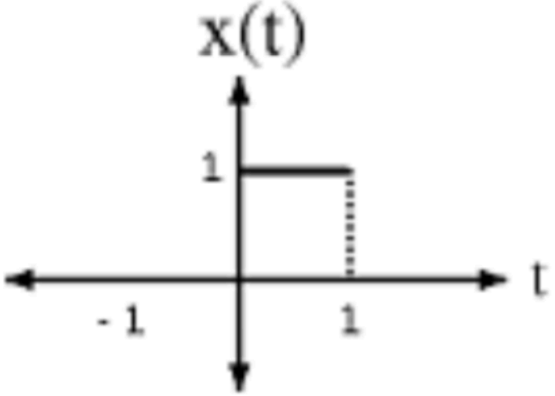
Associative Property (cont.)

- By using the commutative property together with the associative property, we find another very important property of LTI systems.
- From fig(a) and (b) we can conclude that the impulse response of the cascade of two LTI system is the convolution of their individual impulse responses.
- Since convolution is commutative we can compute this convolution of $h_1[n]$ and $h_2[n]$ in either order.

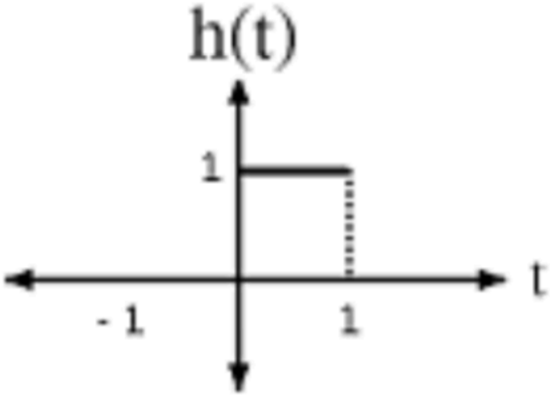
Examples

Example #3

- Convolve the two continuous time signals:



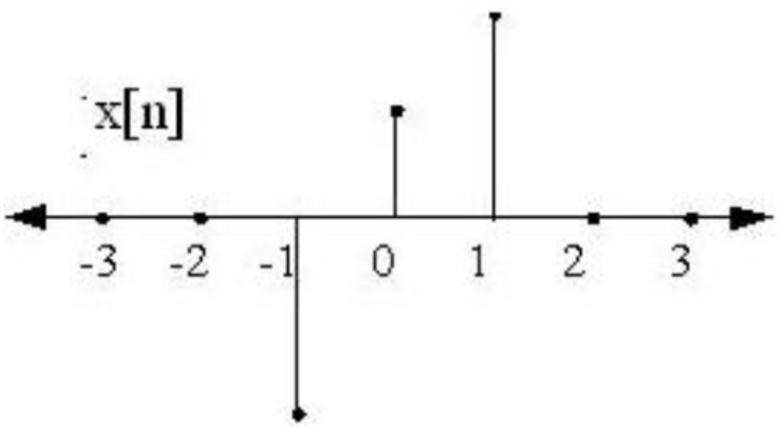
(a)

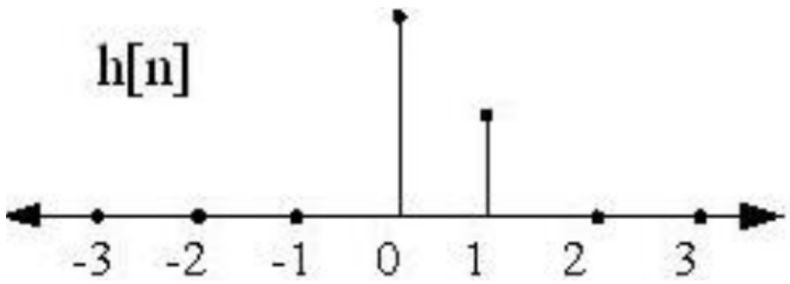


(b)

Example #4

- Convolve the two discrete time signals:





Thank You!