Why do we use Fourier Series?

→ F.S is just a means to represent a periodic signal as an infinite sum of sine wave components.

→ The main reason to use F.S is that we ~~...~~ can better analyse a signal in another domain rather in the original ~~...~~ domain.

D.T Fourier Series:-

→ When k is changed by any integer multiple of N, the identical sequence is generated.

DETERMINATION OF DISCRETE-TIME FOURIER SERIES COEFFICIENTS

→ We have a sequence $x[n]$ that is periodic with fundamental period N.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

→ Multiply both sides by $e^{-j\omega_0 mn}$:-

$$e^{-j\omega_0 mn} x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} e^{-j\omega_0 mn}$$

→ Take summation on $n = \langle N\rangle$:

$$\sum_{n=\langle N\rangle} e^{-j\omega_0 mn} x[n] = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-m)\omega_0 n}$$

$$\sum_{n=\langle N\rangle} e^{-j\omega_0 mn} x[n] = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-m)\omega_0 n}$$

→ Use the sum of geometrical series rule i.e;

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N & a=1 \\ \dfrac{1-a^N}{1-a} & a\neq 1 \end{cases}$$

→ $\therefore \sum_{n=\langle N\rangle} e^{-j\omega_0 mn} x[n] = \sum_{k=\langle N\rangle} a_k N\delta[k-m] \Rightarrow Na_m$

Day/Date

$\rightarrow \quad a_m = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j\omega_0 m n}$

$\rightarrow \quad a_k = a_{k+N}$

EXAMPLE #1:-

$$x[n] = \sin \omega_0 n$$

SOL:-

→ This is the discrete-time counterpart of the signal

$$x(t) = \sin \omega_0 t.$$

→ For the case when $2\pi/\omega_0$ is an integer N, that is when

$$\omega_0 = \frac{2\pi}{N}$$

→ $x[n]$ is periodic with fundamental period N. and we obtain a result that is exactly analogous to the continuous time case.

→ Expanding the signal as a sum of two complex exponentials, we get.

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$

→ we see by inspection

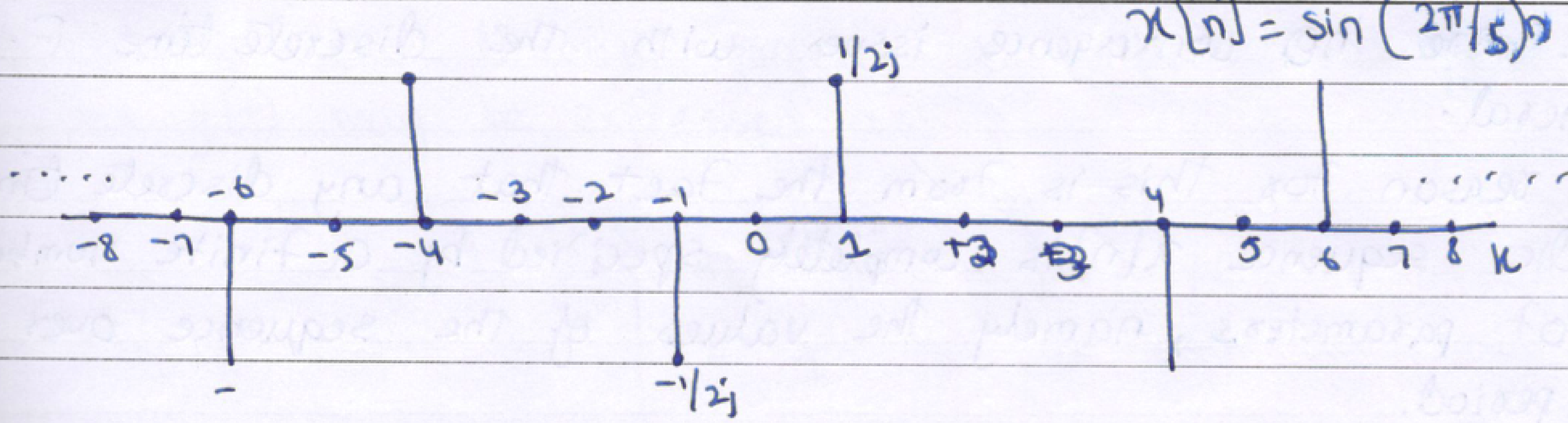
$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}$$

→ remaining coefficients over the interval of summation are zero.

→ As described earlier, these coefficients repeat with period N, thus $a_{N+1} = \frac{1}{2j}$ and $a_{N-1} = -\frac{1}{2j}$.

→ For example:- F.S coefficients for this example with $N=5$ are shown below.

Date



→ They repeat periodically.

EXAMPLE # 2:

$x[n] = \sin(2\pi n/3)$ is periodic with fundamental ~~frequency~~ period $N_o = 3$. Calculate the DTFS coefficients.

Sol:-

$$\sin\left(\frac{2\pi n}{3}\right) = \frac{1}{2j} e^{j2\pi n/3} - \frac{1}{2j} e^{-j2\pi n/3}$$

As $x[n] = \sum_{m=\langle 3\rangle} X_m e^{jm\frac{2\pi}{3}n}$

Choosing index range $\langle 3\rangle = -1, 0, 1$.

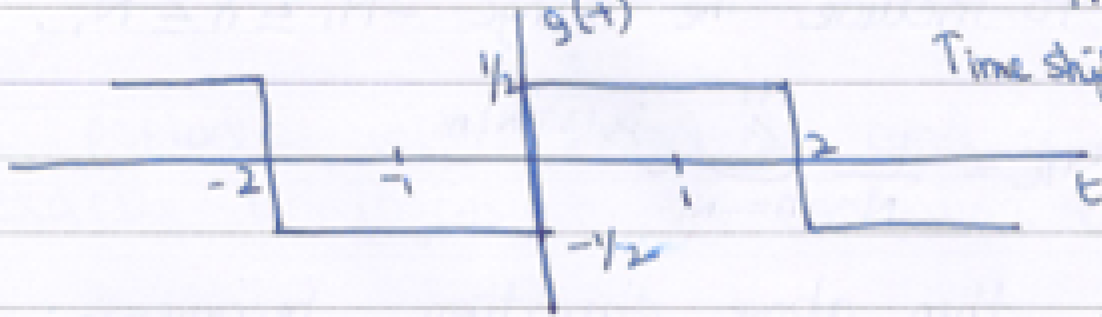
$X_{-1} = -\frac{1}{2j}$, $X_0 = 0$, $X_1 = \frac{1}{2j}$

EXAMPLE # 3:-

Linearity:-

$A x_1(t) + B x_2(t) \Leftrightarrow A a_k + B b_k$

Time shift:- $x(t-t_0) \Leftrightarrow a_k e^{-jk\omega_0 t_0}$



Sol:-

$T = 4$ & $T_1 = 1$

$$g(t) = x(t-1) - \frac{1}{2} \rightarrow (1)$$

→ According to time shift property, if Fourier Series coefficients of $x(t)$ are denoted by a_k, the F.S coefficients of $x(t-1)$ may be expressed as:-

$$b_k = a_k e^{-jk\pi/2}$$

$\therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} \Rightarrow \pi/2$

→ The Fourier series coefficients of the dc offset in $g(t)$ i-e the term $-1/2$ on the right hand side of equ (1) are given by:-

$$c_k = \begin{cases} 0 & \text{for } k \neq 0 \\ -1/2 & \text{for } k = 0. \end{cases}$$

⇒ Applying the linearity property we conclude that the coefficients for $g(t)$ may be expressed as:-

$$d_k = \begin{cases} a_k e^{-jk\pi/2} & \text{for } k \neq 0 \\ a_0 - \frac{1}{2} & \text{for } k = 0 \end{cases}$$

where each a_k may now be replaced by ~~the~~ :-

$$a_k = \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}$$, then we have

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2} & \text{for } k \neq 0 \\ 0 & \text{for } k = 0 \end{cases}$$

Day/Date MONDAY / 14th NOV, 16 LECTURE #9

EXAMPLE #1:-

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

$$h(t) = e^{-t} u(t)$$

SOL:-

→ To calculate the Fourier series coefficients of the output y(t) we first compute the frequency response:-

$$H(j\omega) = \int_0^\infty e^{-\tau} e^{-j\omega\tau} \, d\tau = \int_0^\infty e^{-\tau + (-j\omega\tau)} \, d\tau$$

$$= \int_0^\infty e^{-\tau - j\omega\tau} \, d\tau = \int_0^\infty e^{-\tau(1+j\omega)} \, d\tau$$

$$= \frac{1}{1+j\omega} e^{-\tau} \Big|_0^\infty = -\frac{1}{1+j\omega} \left[e^{-\infty} - e^{-0} \right]$$

$$H(j\omega) = -\frac{1}{1+j\omega} \left[-e^0 \right] \Rightarrow \frac{1}{1+j\omega}$$

→ The output is:-

$$y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t}$$

→ where $b_k = a_k H(jk\omega_0) = a_k H(jk2\pi)$, so that

$$b_0 = 1, \quad b_1 = \frac{1}{4}\left[\frac{1}{1+j2\pi}\right], \quad b_{-1} = \frac{1}{4}\left[\frac{1}{1-j2\pi}\right]$$

$$b_2 = \frac{1}{2}\left[\frac{1}{1+j4\pi}\right], \quad b_{-2} = \frac{1}{2}\left[\frac{1}{1-j4\pi}\right]$$

$$b_3 = \frac{1}{3}\left[\frac{1}{1+j6\pi}\right], \quad b_{-3} = \frac{1}{3}\left[\frac{1}{1-j6\pi}\right]$$

Exercise Problems:-

Problem #1:-

Express $x(t)$ in the form:-

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

Sol:-

$x(t)$ is real valued.

$T = 8$, $a_1 = a_{-1} = 2$, $a_3 = a^*_{-3} = 4j$

→ Using the Fourier series synthesis equ:-

$$x(t) = a_1 e^{j(2\pi/T)t} + a_{-1} e^{-j(2\pi/T)t} + a_3 e^{j3(2\pi/T)t} + a_{-3} e^{-j3(2\pi/T)t}$$

$$= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4j e^{j3(2\pi/8)t} - 4j e^{-j3(2\pi/8)t}$$

$$= 2\left[e^{j(\pi/4)t} + e^{-j(\pi/4)t}\right] + 4j\left[e^{j(6\pi/8)t} - e^{-j(6\pi/8)t}\right]$$

$$= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{6\pi}{8}t\right)$$

$$x(t) \Rightarrow 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)$$

PROBLEM #9:-

$x[n]$ = real and odd periodic signal

$N = 7$, $a_k = ?$

$a_{15} = j$, $a_{16} = 2j$, $a_{17} = 3j$

$a_0 = ?$ $a_{-1} = a_{-2} = a_{-3} = ?$

SOL:-

Since the Fourier series coefficients repeat every N, we have

$$a_1 = a_{15} \quad , \quad a_2 = a_{16} \quad , \quad \& \quad a_3 = a_{17}$$

→ Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd.
Therefore, $a_0 = 0$ and.

$$a_1 = -a_{-1} \quad , \quad a_2 = -a_{-2} \quad , \quad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j \quad , \quad a_{-2} = -2j \quad , \quad a_{-3} = -3j$$

PROBLEM #5:-

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \frac{\sin(4\omega)}{\omega}$$

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

$T = 8$, $y(t) = ?$

SOL:-

→ let us first evaluate the Fourier series coefficients of $x(t)$.

→ Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd.

→ Therefore, $a_0 = 0$. Now,

$$a_k = \frac{1}{8}\int_{0}^{8} x(t)e^{-j(2\pi/8)kt}\,dt$$

$$= \frac{1}{8}\int_0^4 e^{-j(2\pi/8)kt}\,dt - \frac{1}{8}\int_4^8 e^{-j(2\pi/8)kt}\,dt$$

$$a_k = \frac{1}{j\pi k}\left[1 - e^{-j\pi k}\right]$$

→ clearly the above expression evaluates to zero for all even values of k.

→ Therefore,

$$a_k = \begin{cases} 0 & k = 0, \pm 2, \pm 4, \ldots \\ \frac{2}{j\pi k} & k = \pm 1, \pm 3, \pm 5, \ldots \end{cases}$$

→ When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} \Rightarrow \frac{\pi}{4}$.

⇒ since a_k is nonzero only for odd values of k, we need to evaluate the above summation only for odd k.

→ Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

→ is alway zero for odd values of k. Therefore,

$$y(t) = 0$$

PROBLEM #4:-

$x[n]$ = real & even signal , $N = 10$, a_k

$a_{11} = 5$

$$\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = 50$$

$x[n] = A\cos(Bn + C) = ?$

Sol:-

→ Since the Fourier Series coefficients repeat every $N = 10$, we have $a_1 = a_{11} = 5$.

→ Furthermore, since $x[n]$ is real and even, a_k is also real and even.

→ Therefore, $a_1 = a_{-1} = 5$.

→ We are also given that $\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = 50$

→ Using Parseval's relation,

$$\sum_{k=\langle N\rangle} |a_k|^2 = 50$$

$$\sum_{k=-1}^{8} |a_k|^2 = 50$$

$$|a_{-1}|^2 + |a_1|^2 + |a_0|^2 + \sum_{k=2}^{8} |a_k|^2 = 50$$

Day/Date

$$|5|^2 + |5|^2 + a_0^2 + \sum_{k=2}^{8} |a_k|^2 = 50$$

$$a_0^2 + \sum_{k=2}^{8} |a_k|^2 = 0$$

→ There fore, $a_k = 0$ for $k = 2, \ldots\ldots, 8$.

→ Now using the synthesis equation, we have.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-1}^{8} a_k e^{j\frac{2\pi}{10}kn}$$

$$= 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n}$$

$$x[n] \Rightarrow 10\cos\left(\frac{\pi}{5}n\right)$$