Signal & Systems
Lecture #8

08th May 18

Fourier Series

Discrete-Time Periodic Signals

Fourier Series Representation of DT

- The Fourier series representation of discrete-time periodic signal is finite as opposed to the infinite series representation required for continuous-time periodic signals.

Linear Combinations of Harmonically Related Complex Exponentials

- A discrete-time signal x[n] is periodic with period N if: x[n] = x[n+N].

- The fundamental period is the smallest positive N and the fundamental frequency is $\omega_0 = \frac{2\pi}{N}$.

- The set of all discrete-time complex exponential signals that are periodic with period N is given by:

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots..$$

- All of these signals have fundamental frequencies that are multiples of 2π/N and thus are harmonically related.

Linear Combinations of Harmonically Related Complex Exponentials (cont.)

- There are only N distinct signals in the set this is because the discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. That is: $\phi_k[n] = \phi_{k+rN}[n]$
- The representation of periodic sequences in terms of linear combinations of the sequences $\Phi_k[n]$ is:

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}$$

- Since the sequences $\Phi_k[n]$ are distinct over a range of N successive values of k, the summation in above equation need include terms over this range.
- Thus the summation is on k as k varies over a range of N successive integers beginning with any value of k.

Linear Combinations of Harmonically Related Complex Exponentials (cont.)

- We indicate this by expressing the limits of the summation as k=<N>. That is:

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

Discrete-Time Fourier Series Coefficients

- Assuming x[n] is square-summable i.e., $\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 < \infty$ or x[n] satisfies the Dirichlet conditions.

- In this case we have:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \qquad \textit{Synthesis} \quad \textit{Equation}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}, \qquad \textit{Analysis} \quad \textit{Equation}$$

- As in continuous time, the discrete-time Fourier series coefficient a_k are often referred to as the spectral coefficients of x[n].

- These coefficients specify a decomposition of x[n] into a sum of N harmonically related complex exponentials.

Example #1

- Consider the signal:

$$x[n] = \sin \omega_0 n$$

- Which is the discrete-time counterpart of the signal .
- x[n] is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers.

Example #2

- The signal $x[n] = \sin(2\pi n/3)$ is periodic with fundamental period $N_0=3$. calculate the DTFS coefficients.

Properties of Fourier Series Coefficients

Linearity

- For continuous-time Fourier series, we have:

$$x_1(t) \leftrightarrow a_k \quad and \quad x_2(t) \leftrightarrow b_k$$

$$Ax_1(t) + Bx_2(t) \leftrightarrow Aa_k + Bb_k$$

- For Discrete-time case, we have:

$$x_1(t) \leftrightarrow a_k \quad and \quad x_2(t) \leftrightarrow b_k$$

$$Ax_1(t) + Bx_2(t) \leftrightarrow Aa_k + Bb_k$$

Time Shift

$$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$$

$$x[n-n_0] \leftrightarrow a_k e^{-jk\Omega_0 n_0}$$

- Proof: Let us consider the Fourier series coefficient b_k of the signal $y(t)=x(t-t_o)$.

$$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-j\omega_0 t}\,dt$$

- Letting $\tau = t-t_o$ in the integral, we obtain:

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}\,dt = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}\,dt$$

$$\text{where} \quad x(t) \leftrightarrow a_k. \quad \text{Therefore,}$$

$$x(t-t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$$

Time Reversal

$$x(-t) \leftrightarrow a_{-k}$$

$$x[-n] \leftrightarrow a_{-k}$$

- Proof: Consider a signal y(t) = x(-t). The Fourier series representation of x(-t) is:

$$x(-t) = \sum_{-\infty}^{\infty} a_k e^{-jk2\pi t/T}$$

- Letting k = -m, we have:

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

- Thus:

$$x(-t) \leftrightarrow a_{-k}$$

Time Scaling

- Time scaling is an operation that in general changes the period of the underlying signal.

- Specifically if $x(t)$ is periodic with period T and fundamental frequency $\omega_o=2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_o$.

Properties of Continuous-Time Fourier Series

- **Multiplication:**
  - Suppose that x(t) and y(t) are both periodic with period T and that:

$$x(t) \leftrightarrow a_k$$

$$v(t) \leftrightarrow b_k$$

  - Since the product x(t) y(t) is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for x(t) and y(t). The result is:

$$x(t)y(t) \leftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

  - The sum on the R.H.S may be interpreted as the Discrete-time convolution of the sequence x(t) and y(t).

Properties of Continuous-Time Fourier Series (cont.)

- **Conjugation and Conjugate Symmetry:**
  - Real $x(t) \leftrightarrow a_{-k} = a_k^*$ (conjugate symmetric)
  - Real & Even $x(t) \leftrightarrow a_k = a_k^*$ (real and even a_k)
  - Real & Odd $x(t) \leftrightarrow a_k = -a_k^*$ (purely imaginary and odd a_k), $a_0=0$
  - Even part of $x(t) \leftrightarrow \mathrm{Re}\{a_k\}$
  - Odd part of $x(t) \leftrightarrow j\,\mathrm{Im}\{a_k\}$

Properties of Continuous-Time Fourier Series (cont.)

- **Parseval's Relation:**
  - Parseval's relation for continuous-time periodic signal is:

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

  - Where a_k are the Fourier series coefficients of x(t) and T is the period of the signal.
  - L.H.S of the above equation is the average power (i.e., energy per unit time) in one period of the periodic signal x(t).
  - Also:

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2\,dt = |a_k|^2$$

  - So that $|a_k|^2$ is the average power in the kth harmonic component of x(t).
  - Thus Parseval's relation states that the total average power in a periodic signals equals the sum of the average powers in all of its harmonic components.

Properties of Discrete-Time Fourier Series

- **Multiplication:**
  - In discrete-time, suppose that: $x[n] \leftrightarrow a_k$

    and

    $y[n] \leftrightarrow b_k$
  - Are both periodic with period N. then the product x[n] y[n] is also periodic with period N.
  - Its Fourier coefficients d_k are given by:

$$x[n]y[n] \leftrightarrow d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

  - The result is a periodic convolution between the FS sequences.
  - w[n] is periodic with N.

Properties of Discrete-Time Fourier Series (cont.)

- **<u>First Difference:</u>**
  - If x[n] is periodic with period N, then so is y[n], since shifting x[n] or linearly combining x[n] with another periodic signal whose period is N always results in a periodic signal with period N.
  - Also, if:

$$x[n] \leftrightarrow a_k$$

  - Then the Fourier coefficients corresponding to the first difference of x[n] may be expressed as:

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-jk(2\pi/N)}\right) a_k$$

Properties of Discrete-Time Fourier Series (cont.)

- **<u>Parseval's Relation:</u>**
  - Parseval's relation for discrete-time periodic signals is given by:

$$\frac{1}{N}\sum_{n=\langle N\rangle}\left|x[n]\right|^2=\sum_{n=\langle N\rangle}\left|a_k\right|^2$$

  - The average power in a periodic signal = the sum of the average power in all of its harmonic components.

Fourier Series & LTI Systems

Fourier Series & LTI Systems

- The response of a continuous-time LTI system with impulse response h(t) to a complex exponential signal e^{st} is the same complex exponential multiplied by a complex gain:

$$y(t) = H(s)e^{st}$$

$$where$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$$

- In particular, for s=jω, the output is $y(t)=H(j\omega)e^{j\omega t}$.

- The complex functions H(s) and H(jω) are called the system function (or transfer function) and the frequency response, respectively.

Fourier Series & LTI Systems (cont.)

- By superposition, the output of an LTI system to a periodic signal represented by a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad is \quad given \quad by$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

- That is, the Fourier series coefficients b_k of the periodic output y(t) are given by:

$$b_k = a_k H(jk\omega_0)$$

- Similarly, for discrete time signals and systems, response h[n] to a complex exponential signal $e^{j\omega n}$ is the same complex exponential multiplied by a complex gain:

Fourier Series & LTI Systems (cont.)

$$y[n] = H(jk\omega_0)e^{jk\omega_0 n}$$

where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Example #4

- Suppose that the periodic signal $x(t)=\sum_{k=-3}^{3} a_k e^{jk2\pi t}$ with $a_0=1$, $a_1=a_{-1}=1/4$, $a_2=a_{-2}=1/2$, and $a_3=a_{-3}=1/3$ is the input signal to an LTI system with impulse response $h(t)=e^{-t}u(t)$.

Exercise Problems

Problem #1

- A continuous-time periodic signal x(t) is real valued and has a fundamental period T=8. The non-zero Fourier series coefficients for x(t) are:

$$a_1 = a_{-1} = 2, a_3 = a^*_{-3} = 4j$$

- Express x(t) in the form:

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

Problem #2

- Let x[n] be a real and odd periodic signal with period N=7 and Fourier coefficients a_k. Given that:

$$a_{15} = j, a_{16} = 2j, a_{17} = 3j$$

- Determine the values of a_0, a_{-1}, a_{-2} and a_{-3} .

Problem #3

- Suppose we are given the following information about a signal x[n]:
  - 1. x[n] is a real and even signal.
  - 2. x[n] has period N=10 and Fourier coefficients a_k .
  - 3. $a_{11}=5$
  - 4. $\frac{1}{10}\sum_{n=0}^{9}\left|x[n]\right|^2 = 50$
  - Show that x[n] = Acos (Bn+C) , and specify numerical values for the constants A, B and C.

Thank You!