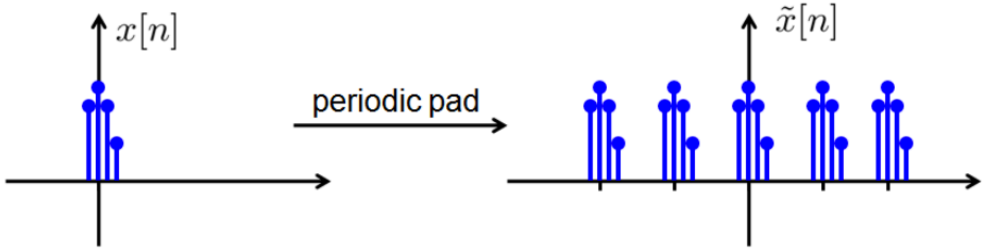
Signal & Systems
Lecture #9

15th May 18

Discrete Time Fourier Transform

Development of the DTFT

- In deriving discrete-time Fourier Transform we have three key steps:
- Step#1:
  - Consider an aperiodic discrete-time signal x[n]. We pad x[n] to construct a periodic signal x'[n].



- Step#2:
  - Since x'[n] is periodic, by discrete-time Fourier series we have:

$$x'[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

Development of the DTFT (cont.)

- Where a_k is:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x'[n]e^{-jk(2\pi/N)n}$$

- Here, $\omega_o = 2\pi/N$.
- Now note that x'[n] is a periodic signal with period N and the non-zero entries of x'[n] in a period are the same as the non-zero entries of x[n].
- Therefore, it holds that:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x'[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$$

Development of the DTFT (cont.)

- If we define: $$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$
- Then: $$a_k=\frac{1}{N}\sum_{n=-\infty}^{\infty}x[n]e^{-jk(2\pi/N)n}=\frac{1}{N}X\left(e^{jk\omega_0}\right)$$
- Step#3:
  - Putting above equation in discrete-time Fourier series equation, we have:

$$x'[n]=\sum_{k=\langle N\rangle}a_k e^{jk\omega_0 n}$$

$$=\sum_{k=\langle N\rangle}\left[\frac{1}{N}X\left(e^{jk\omega_0}\right)\right]e^{jk\omega_0 n}$$

$$=\frac{1}{2\pi}\sum_{k=\langle N\rangle}X\left(e^{jk\omega_0}\right)e^{jk\omega_0 n}\omega_0,\quad \omega_0=\frac{2\pi}{N}$$

Development of the DTFT (cont.)

- As N→ ∞, ω_0 →0, so the area becomes infinitesimal small and sum becomes integration and x'[n]=x[n],so above equation becomes,

$$x'[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0 \rightarrow \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega$$

- Hence, the Discrete time Fourier transform pair:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Development of the DTFT (cont.)

- The first equation is referred to as synthesis equation and second one as analysis equation.
- $X(e^{j\omega})$ is referred to as the spectrum of $x[n]$.

Example #1

- Consider the signal: $x[n] = a^n u[n], \quad |a| < 1$

- Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

Example #2

- Consider the signal: $x[n] = a^{|n|}, \quad |a| < 1$
- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

- Let m=-n in the first summation we obtain,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{m=1}^{\infty} a^{m} e^{j\omega m}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{n} + \sum_{m=1}^{\infty} \left(ae^{j\omega}\right)^{m}$$

Example #2 (cont.)

- Both of these summations are infinite geometric series that we can evaluate in closed form, yielding:

$$X\left(e^{j\omega}\right) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{1-a^2}{1-2a\cos\omega+a^2}$$

The Fourier Transform for Periodic Signals

Periodic Signals

- For a periodic discrete-time signal: $x[n] = e^{j\omega_0 n}$
- The discrete-time Fourier transform must be periodic in ω with period 2π.
- Then the Fourier transform of x[n] should have impulses at ω_o , $\omega_o \pm 2\pi$, $\omega_o \pm 4\pi$, and so on.
- In fact, the Fourier transform of x[n] is the impulse train:

$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Now consider a periodic sequence x[n] with period N and with the Fourier series representation:

Periodic Signals (cont.)

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

- In this case, the Fourier transform is:

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- So that the Fourier transform of a periodic signal can directly constructed from its Fourier coefficients.

Example #3

- Consider the periodic signal:

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad where \quad \omega_0 = \frac{2\pi}{5}$$

- Solution:
  - From the equation of periodicity we can write:

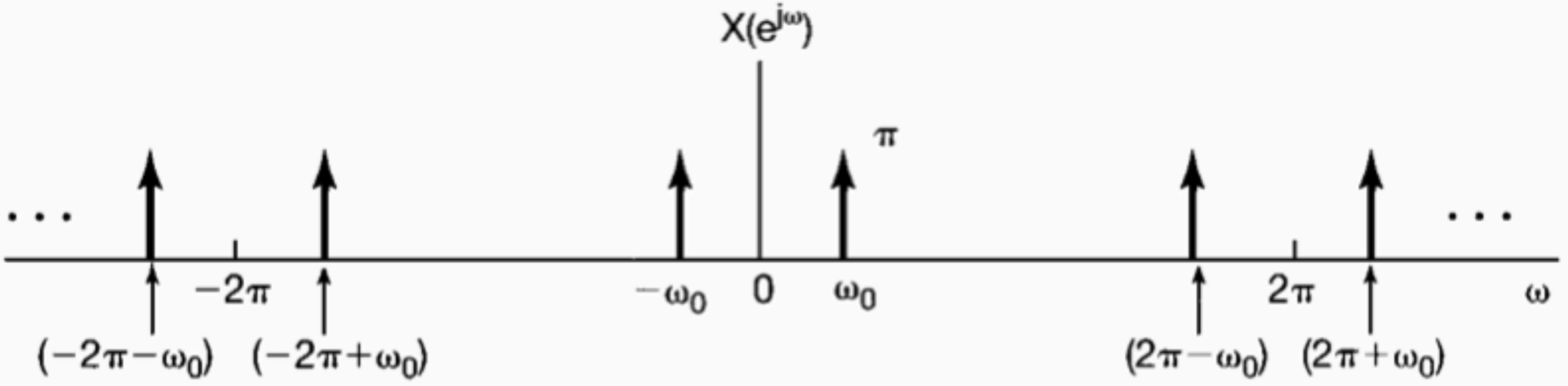
$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} \pi\delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(\omega + \frac{2\pi}{5} - 2\pi l)$$

  - That is,

$$X\left(e^{j\omega}\right) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \le \omega < \pi$$

  - $X(e^{j\omega})$ repeats periodically with a period of 2π, as shown below:

Example #3 (cont.)



Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

Properties of DT Fourier Transform

Periodicity

- The discrete-time Fourier transform is always periodic in ω with period 2π, i.e.,

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

Linearity

- If:

$$x_1[n] \leftrightarrow X_1\left(e^{j\omega}\right)$$

$$And$$

$$x_2[n] \leftrightarrow X_2\left(e^{j\omega}\right)$$

- Then:

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

Time Shifting & Frequency Shifting

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[n-n_0] \stackrel{F}{\leftrightarrow} e^{-j\omega_0 n} X\left(e^{j\omega}\right)$$

$$and$$

$$e^{j\omega_0 n} x[n] \stackrel{F}{\leftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Conjugation & Conjugate Symmetry

- If: $$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$
- Then:

$$x^*[n] \overset{F}{\leftrightarrow} X^*\left(e^{-j\omega}\right)$$

- If x[n] is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is:

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

- From this, it follows that $\text{Re}\left\{X\left(e^{j\omega}\right)\right\}$ is an even function of ω and $\text{Im}\left\{X\left(e^{j\omega}\right)\right\}$ is an odd function of ω.
- Similarly the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

Conjugation & Conjugate Symmetry (cont.)

- Furthermore,

$$Ev\left\{x[n]\right\} \overset{F}{\leftrightarrow} \mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\}$$

and

$$Od\left\{x[n]\right\} \overset{F}{\leftrightarrow} j\,\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\}$$

Differencing & Accumulation

- If: $x[n] \leftrightarrow X(e^{j\omega})$
- Then:

$$x[n]-x[n-1]\overset{F}{\longleftrightarrow}\left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

$$For \quad signal,$$

$$y[n]=\sum_{m=-\infty}^{n} x[m],$$

- Its Fourier transform is given as:

$$\sum_{m=-\infty}^{n} x[m]\overset{F}{\longleftrightarrow}\frac{1}{\left(1-e^{-j\omega}\right)}X(e^{j\omega})+\pi X(e^{j0})\sum_{m=-\infty}^{+\infty}\delta(\omega-2\pi k)$$

Time Reversal

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$x[-n] \overset{F}{\longleftrightarrow} X\left(-e^{j\omega}\right)$$

Differentiation in Frequency

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's Relation

- If:

$$x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

- Then:

$$\sum_{n=-\infty}^{+\infty} \left|x[n]\right|^2 = \frac{1}{2\pi} \int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Convolution Property

- If x[n], h[n] and y[n] are the input, impulse response, and output respectively, of an LTI system, so that,

$$y[n] = x[n] * h[n]$$

$$then,$$

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$

- Where $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of x[n], h[n] and y[n] respectively.

Example #4

- Consider an LTI system with impulse response:

$$h[n] = \delta[n - n_0]$$

- The frequency response is:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

- Thus for any input x[n] with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is:

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Multiplication Property

- It states that:

$$y[n] = x_1[n]x_2[n] \overset{F}{\longleftrightarrow} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\omega}\right)X_2\left(e^{j(\omega-\theta)}\right)d\theta$$

Example #5

- Consider the signal: $x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$
- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \delta[n+1]\right) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1] e^{-j\omega n}$$

$$X(e^{j\omega}) = 1 + e^{-j\omega} + e^{j\omega} = 1 + 2\cos\omega$$

Systems Characterized by Linear Constant-Coefficient Difference Equations

Linear Constant-Coefficient Difference Equations

- A general linear constant-coefficient difference equation for an LTI system with input x[n] and output y[n] is of the form,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Which is usually referred to as Nth-order difference equation.

- If $x[n] = e^{j\omega n}$ is the input to an LTI system, then the output must be of the form $H(e^{j\omega})e^{j\omega n}$. Substituting these expressions into above equation and performing some algebra allow us to solve for $H(e^{j\omega})$.

- Based on convolution, above equation can be written as:

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

Linear Constant-Coefficient Difference Equations (cont.)

- Applying the Fourier transform to both sides and using the linearity and time-shifting properties we obtain the following expression:

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y\left(e^{j\omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\omega} X\left(e^{j\omega}\right)$$

- Or equivalently

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

- The frequency response of the LTI system can be written down by inspection as well.

Example #6

- Consider the causal LTI system that is characterized by the difference equation:

$$y[n]-ay[n-1]=x[n],\quad |a|<1$$

- The frequency response of this system is:

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{1}{1-ae^{-j\omega}}$$

- The impulse response is given by:

$$h[n]=a^{n}u[n]$$

Example #7

- Consider the LTI system:

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

- And let the input to this system be:

$$x[n]=\left(\frac{1}{4}\right)^{n}u[n]$$

- Solution:

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)=\left[\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}$$

Example #7 (cont.)

- Using the partial fraction expansion, we get:

$$Y\left(e^{j\omega}\right)=\frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}}+\frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}+\frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}$$

- Solving the partial fraction gives:

$$B_{11}=-4,\quad B_{12}=-2,\quad B_{21}=8$$

- So that:

$$Y\left(e^{j\omega}\right)=-\frac{4}{1-\frac{1}{4}e^{-j\omega}}-\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}+\frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

Example #7 (cont.)

- The inverse transform i.e., y[n] is:

$$y[n] = \left\{ -4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n \right\} u[n]$$

Problems

Problem #1

- Use the Fourier transform analysis equation to calculate the Fourier transforms of the following signals:

$$(a): \quad \left(\frac{1}{2}\right)^{|n-1|}$$

$$(b): \quad \delta[n+2]-\delta[n-2]$$

$$(c): \quad \sin\left(\frac{\pi}{2}n\right)+\cos(n)$$

Problem #2

- Consider a causal and stable LTI system S whose input x[n] and output y[n] are related through the second-order difference equation:

$$y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]$$

- (a): Determine the frequency response $H(e^{j\omega})$ for the system S.
- (b): Determine the impulse response h[n] for the system S.

Thank You!