Fourier Transform:-

Example #1:-

$$x[n] = a^n u[n] \quad , |a| < 1$$

Sol:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

Using Geometric series formula:- $S = \sum_{k=0}^{\infty} r^k \Rightarrow \frac{1}{1-r}$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$\Rightarrow$ Graph is shown in slides. All of these functions are periodic in ω with period 2π.

Example #2:-

$$x[n] = a^{|n|} \quad , |a| < 1$$

Sol:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$m = -(-\infty)$

Substitute $m = -n$ in first summation

$$X(e^{j\omega}) = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{m=1}^{\infty} (a e^{j\omega})^m + \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

Using geometric series, we have.

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$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{(1-ae^{j\omega}) + ae^{j\omega}(1-ae^{-j\omega})}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

$$= \frac{1-\cancel{ae^{j\omega}} + \cancel{ae^{j\omega}} - a^2 e^{j\omega-j\omega}}{1-ae^{j\omega}-ae^{-j\omega}+a^2e^{-j\omega+j\omega}} \quad \therefore e^{j\omega-j\omega} = e^0 \Rightarrow 1$$

$$X(e^{j\omega}) = \frac{1-a^2}{1+a^2-2a\cos\omega}$$

EXAMPLE # 3:-

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} \quad , \omega_0 = \frac{2\pi}{5}$$

SOL:-

Using equation of periodicity: $X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$

$$X(e^{j\omega}) = \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\omega-\frac{2\pi}{5}-2\pi l\right) + \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta\left(\omega+\frac{2\pi}{5}-2\pi l\right)$$

$$= \frac{1}{\cancel{2}}\times\cancel{2}\sum_{l=-\infty}^{\infty} \pi\delta\left(\omega-\frac{2\pi}{5}-2\pi l\right) + \frac{1}{\cancel{2}}\times\cancel{2}\sum_{l=-\infty}^{\infty} \pi\delta\left(\omega+\frac{2\pi}{5}-2\pi l\right)$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\delta\left(\omega-\frac{2\pi}{5}-2\pi l\right) + \sum_{l=-\infty}^{\infty} \pi\delta\left(\omega+\frac{2\pi}{5}-2\pi l\right)$$

That is:-

$$X(e^{j\omega}) = \pi\delta\left(\omega-\frac{2\pi}{5}\right) + \pi\delta\left(\omega+\frac{2\pi}{5}\right) \quad , \quad -\pi \le \omega < \pi$$

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EXAMPLE #4:-

$$h[n] = \delta[n-n_0].$$

SOL:-

The frequency response is:-

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n-n_0)e^{-j\omega n} \Rightarrow e^{-j\omega n_0}$$

→ Thus for any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is:

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

EXAMPLE #5:-

$$x[n] = \delta[n] + \delta[n-1] + \delta[n+1].$$

SOL:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\delta[n] + \delta[n-1] + \delta[n+1]\right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n-1]e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta[n+1]e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{j\omega} \Rightarrow 1 + 2\cos\omega$$

EXAMPLE #1:-

$$y[n] - a y[n-1] = x[n] \; , \; |a| < 1 \rightarrow ①$$

SOL:-

The frequency response : $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1}{1-ae^{-j\omega}}$

$y[n] \Rightarrow Y(e^{j\omega})$

$y[n-1] \Rightarrow e^{-j\omega} Y(e^{j\omega})$ ∴ using time Shifting property

$x[n] \Rightarrow X(e^{j\omega})$

Hence equ ① becomes.

$$Y(e^{j\omega}) - ae^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega}) [1 - ae^{-j\omega}] = X(e^{j\omega})$$

Cross multiplying gives

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \Rightarrow \frac{1}{1-ae^{-j\omega}}$$

We know that it is the Fourier transform of $a^n u[n]$. Hence the impulse response of the system is :-

$$h[n] = a^n u[n].$$

EXAMPLE #2:-

$$y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n]$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$Y(e^{j\omega}) = ? \; , \; y[n] = ?$$

SOL:-

$$Y(e^{j\omega}) - \frac{3}{4} e^{-j\omega} Y(e^{j\omega}) + \frac{1}{8} e^{-2j\omega} Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}\right] = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \Rightarrow \frac{2}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}}$$

$$-\frac{1}{4}-\frac{1}{2} \Rightarrow \frac{-1-2}{4} \Rightarrow -\frac{3}{4}$$

Factor the denominator of $H(e^{j\omega})$ i.e.,

$$1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega} = \left[1-\frac{1}{2}e^{-j\omega}\right]\left[1-\frac{1}{4}e^{-j\omega}\right]$$

$$H(e^{j\omega}) = \frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{j\omega}\right)}$$

No need in this example just for information

Using partial fraction method

$$H(e^{j\omega}) = \frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)} = \frac{A}{\left(1-\frac{1}{2}e^{-j\omega}\right)} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

let $s = e^{-j\omega}$

$$\frac{2}{\left(1-\frac{1}{2}s\right)\left(1-\frac{1}{4}s\right)} = \frac{A}{1-\frac{1}{2}s} + \frac{B}{1-\frac{1}{4}s}$$

Cross multiplication gives

$$2 = A\left(1-\frac{1}{4}s\right) + B\left(1-\frac{1}{2}s\right)$$

put $1-\frac{1}{4}s = 0$, $1-\frac{1}{2}s = 0$

$$\frac{1}{4}s = 1 \qquad \frac{1}{2}s = 1$$

$$s = 4 \qquad s = 2$$

when $s = 4$.

$$2 = A\left[1-\frac{1}{\cancel{4}}(\cancel{4})\right] + B\left[1-\frac{1}{\cancel{2}}(\cancel{4})^{2}\right]$$

$$2 = A[0] + B[1-2]$$

$$2 = -B$$

$$B \Rightarrow -2$$

ate

just for information

when $s = 2$

$$2 = A\left[1 - \frac{1}{\cancel{4}_2}(\cancel{2})\right] + B\left[1 - \frac{1}{\cancel{2}}(\cancel{2})\right]$$

$$2 = A\left[1 - \frac{1}{2}\right]$$

$$2 = A\left[\frac{2-1}{2}\right]$$

$$2 = \frac{A}{2} \Rightarrow A = 4$$

Hence,

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

~~Hen~~ Now, $H(e^{j\omega}) = \dfrac{2}{\left[1 - \frac{1}{2}e^{-j\omega}\right]\left[1 - \frac{1}{4}e^{-j\omega}\right]}$

$$x[n] = \left[\frac{1}{4}\right]^n u[n] \qquad \because a^n u[n] = \frac{1}{1 - ae^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$= \left[\frac{2}{\left[1 - \frac{1}{2}e^{-j\omega}\right]\left[1 - \frac{1}{4}e^{-j\omega}\right]}\right]\left[\frac{1}{1 - \frac{1}{4}e^{-j\omega}}\right]$$

$$Y(e^{j\omega}) = \frac{2}{\left[1 - \frac{1}{2}e^{-j\omega}\right]\left[1 - \frac{1}{4}e^{-j\omega}\right]^2}$$

Using partial fraction method :-

$$Y(e^{j\omega}) = \frac{2}{\left[1 - \frac{1}{2}e^{-j\omega}\right]\left[1 - \frac{1}{4}e^{-j\omega}\right]^2} = \frac{A}{\left[1 - \frac{1}{4}e^{-j\omega}\right]} + \frac{B}{\left[1 - \frac{1}{4}e^{-j\omega}\right]^2} + \frac{C}{1 - \frac{1}{2}e^{-j\omega}}$$

Cross multiplication gives

$$2 = A\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right) + B\left(1-\frac{1}{2}e^{-j\omega}\right) + C\left(1-\frac{1}{4}e^{-j\omega}\right)^2$$

let $e^{-j\omega} = s$

$$2 = A\left[1-\frac{1}{2}s\right]\left[1-\frac{1}{4}s\right] + B\left[1-\frac{1}{2}s\right] + C\left[1-\frac{1}{4}s\right]^2$$

let $s=2$, $s=4$

when $s=2$,

$$2 = A\left[1-\frac{1}{2}(2)\right]\left[1-\frac{1}{4}(2)\right] + B\left[1-\frac{1}{2}(2)\right] + C\left[1-\frac{1}{4}(2)\right]^2$$

$$2 = A(0)\left(1-\frac{1}{2}\right) + B(0) + C\left[1-\frac{1}{2}\right]^2$$

$$2 = C\left[\frac{2-1}{2}\right]^2 \Rightarrow 2 = C\left[\frac{1}{4}\right]$$

$$C \Rightarrow 8$$

when $s=4$,

$$2 = A\left[1-\frac{1}{2}(4)\right]\left[1-\frac{1}{4}(4)\right] + B\left[1-\frac{1}{2}(4)\right] + C\left[1-\frac{1}{4}(4)\right]^2$$

$$2 = A(0) + B(1-2) + C(0)$$

$$B \Rightarrow -2$$

for the value of A we will put $s=0$ as we don't have any other value for s from the equation above

Hence, when $s=0$

$$2 = A\left[1-\frac{1}{2}(0)\right]\left[1-\frac{1}{4}(0)\right] + B\left[1-\frac{1}{2}(0)\right] + C\left[1-\frac{1}{4}(0)\right]^2$$

$$2 = A[1][1] + (-2)(1) + (8)(1)$$

$$2 = A - 2 + 8$$

$$2 = A + 6$$

$$A = -6 + 2 \Rightarrow -4$$

So, $Y(e^{j\omega}) = -\dfrac{4}{1-\frac{1}{4}e^{-j\omega}} - \dfrac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \dfrac{8}{1+\frac{1}{2}e^{-j\omega}}$

Using the properties :-

$$a^n u[n] \doteq \frac{1}{1-ae^{-j\omega}} \quad \& \quad \frac{1}{(1-ae^{-j\omega})^2} = (n+1)a^n u$$

The inverse fourier transform $y[n]$ is :-

$$y[n] = \left[-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right] u[n].$$

PROBLEMS:-

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$n = n+1$
$1 = n+1$
$n = 0$
$n+1-1$

PROBLEM #2 :-

a) $\left[\frac{1}{2}\right]^{|n-1|}$

SOL:-

Using analysis equation : $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n-1|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-(n-1)} e^{-j\omega n} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (1/2)^{(n+1)} e^{j\omega n} + \sum_{n=0}^{\infty} (1/2)^{n} e^{-j\omega(n+1)}$$

$$= \left(\frac{1}{2}\right) \frac{1}{1-(1/2)e^{j\omega}} + e^{-j\omega} \frac{1}{1-(1/2)e^{-j\omega}}$$

$$= \frac{(1/2)\left(1-1/2\, e^{-j\omega}\right) + e^{-j\omega}\left(1-1/2\, e^{j\omega}\right)}{\left(1-\frac{1}{2}e^{j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$= \frac{0.5 - 0.25e^{-j\omega} + e^{-j\omega} - 0.5e^{-j\omega+j\omega}}{1-0.5e^{-j\omega} - 0.5e^{j\omega} + 0.25e^{j\omega-j\omega}}$$

$$= \frac{\cancel{0.5} - 0.25e^{-j\omega} + e^{-j\omega} - \cancel{0.5}}{1-0.5e^{-j\omega} - 0.5e^{j\omega} + 0.25}$$

$\therefore \quad -\frac{1}{2}e^{-j\omega} - \frac{1}{2}e^{j\omega} \Rightarrow -\left[\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega}\right] \Rightarrow -\cos\omega$

$$X(e^{j\omega}) \Rightarrow \frac{0.75e^{-j\omega}}{1.25 - \cos\omega}$$

b) $\delta[n+2] - \delta[n-2]$

Sol:-

$$x[n] = \delta[n+2] - \delta[n-2]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [\delta(n+2) - \delta(n-2)] e^{-j\omega n}$$

$$X(e^{j\omega}) = e^{2j\omega} - e^{-2j\omega} \Rightarrow 2j\sin(2\omega)$$

c) $\sin\left(\frac{\pi}{2}n\right) + \cos(n)$

Sol:-

$$x[n] = \sin\left[\frac{\pi}{2}n\right] + \cos[n]$$

$$= \frac{1}{2j}\left[e^{j\pi/2 n} - e^{-j\pi/2 n}\right] + \frac{1}{2}\left[e^{jn} + e^{-jn}\right]$$

$$\omega_0 = \frac{\pi}{2} \qquad \omega_0 = 1$$

$$e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l)$$

Now if $0 \leq \omega < \pi$ then i-e is periodic. (repeats periodically)

$$X(e^{j\omega}) = \frac{1}{2j}\left[2\pi\left\{\delta\left(\omega - \frac{\pi}{2}\right) - \delta\left(\omega + \frac{\pi}{2}\right)\right\}\right] + \frac{1}{2}\left[2\pi\left\{\delta(\omega - 1) + \delta(\omega + 1)\right\}\right]$$

$$= \frac{\cancel{2}\pi}{\cancel{2}j}\left[\delta\left(\omega - \frac{\pi}{2}\right) - \delta\left(\omega + \frac{\pi}{2}\right)\right] + \frac{\cancel{2}\pi}{\cancel{2}}\left[\delta(\omega - 1) + \delta(\omega + 1)\right]$$

$$X(e^{j\omega}) \Rightarrow \frac{\pi}{j}\left[\delta\left(\omega - \frac{\pi}{2}\right) - \delta\left(\omega + \frac{\pi}{2}\right)\right] + \pi\left[\delta(\omega - 1) + \delta(\omega + 1)\right]$$

PROBLEM # 5:-

$$y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]$$

SOL:-

a) $H(e^{j\omega})=?$

$$Y(e^{j\omega})-\frac{1}{6}e^{-j\omega}Y(e^{j\omega})-\frac{1}{6}e^{-j2\omega}Y(e^{-2j\omega})=X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}\right]=X(e^{j\omega})$$

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}\Rightarrow\frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}}$$

$$-\frac{1}{2}+\frac{1}{3} \Rightarrow \frac{-3+2}{6} \Rightarrow \frac{-1}{6}$$

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$$-\frac{1}{2}e^{j\omega} = -1$$
$$e^{-j\omega} = 2$$

$$H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)}$$

b) $h[n] = ?$

Using Partial Fractions:

$$H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}}$$

Cross multiplication we get:

$$\frac{1}{1} = A\left(1+\frac{1}{3}e^{-j\omega}\right) + B\left(1-\frac{1}{2}e^{-j\omega}\right)$$

let $e^{-j\omega} = -3$, $e^{-j\omega} = 2$

when $e^{j\omega} = 3$

$$1 = A\left[1+\frac{1}{3}(-3)\right] + B\left[1-\frac{1}{2}(-3)\right]$$

$$1 = A(0) + B\left[1-\frac{(-3)}{2}\right]$$

$$1 = B\left[\frac{2+3}{2}\right]$$

$$B\left[\frac{+5}{2}\right] = 1 \Rightarrow B = \cdot 2/5$$

when $e^{-j\omega} = 2$

$$1 = A\left[1+\frac{1}{3}(2)\right] + B\left[1-\frac{1}{2}(2)\right]$$

$$1 = A\left[\frac{3+2}{3}\right]$$

$$A = 3/5$$

$$H(e^{j\omega}) = \frac{3/5}{1-\frac{1}{2}e^{-j\omega}} + \frac{2/5}{1+\frac{1}{3}e^{-j\omega}}$$

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taking the inverse Fourier transform:-

$$h[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(\frac{-1}{3}\right)^n u[n]$$