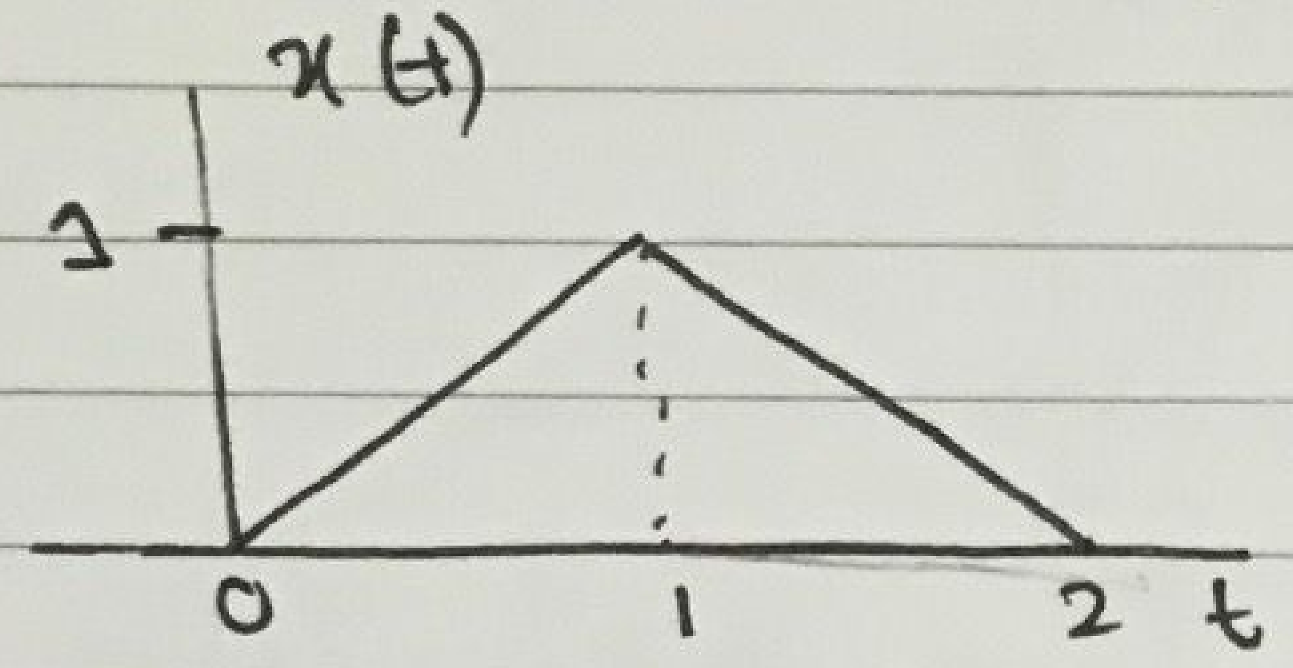
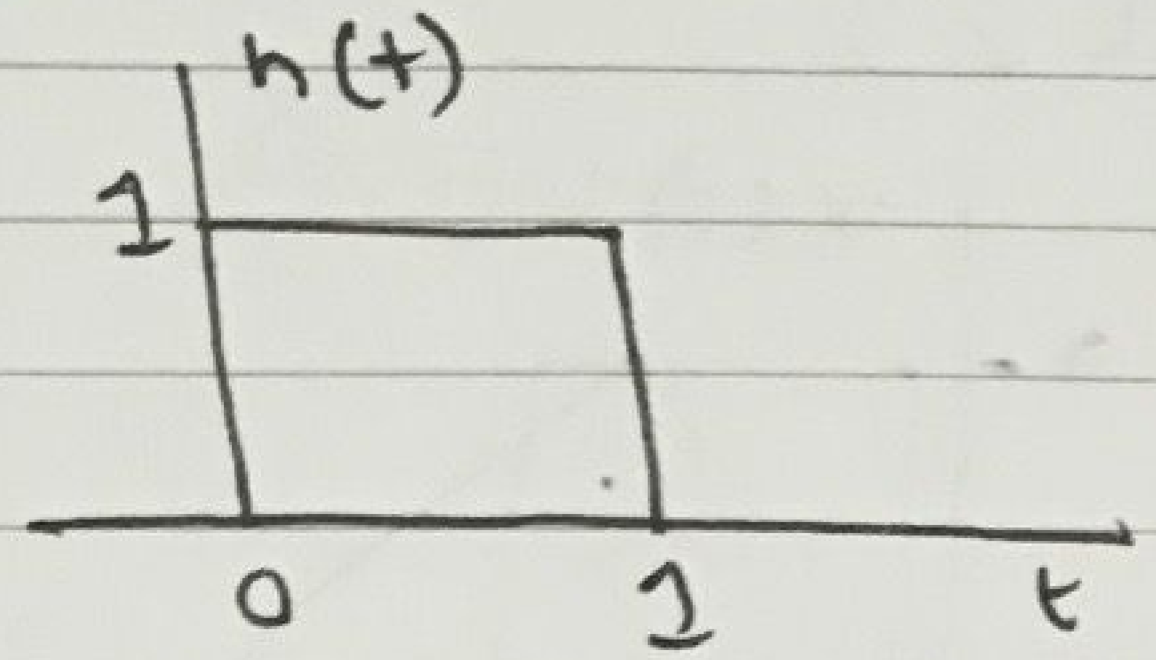
Lecture #11
Revision

Problem #1

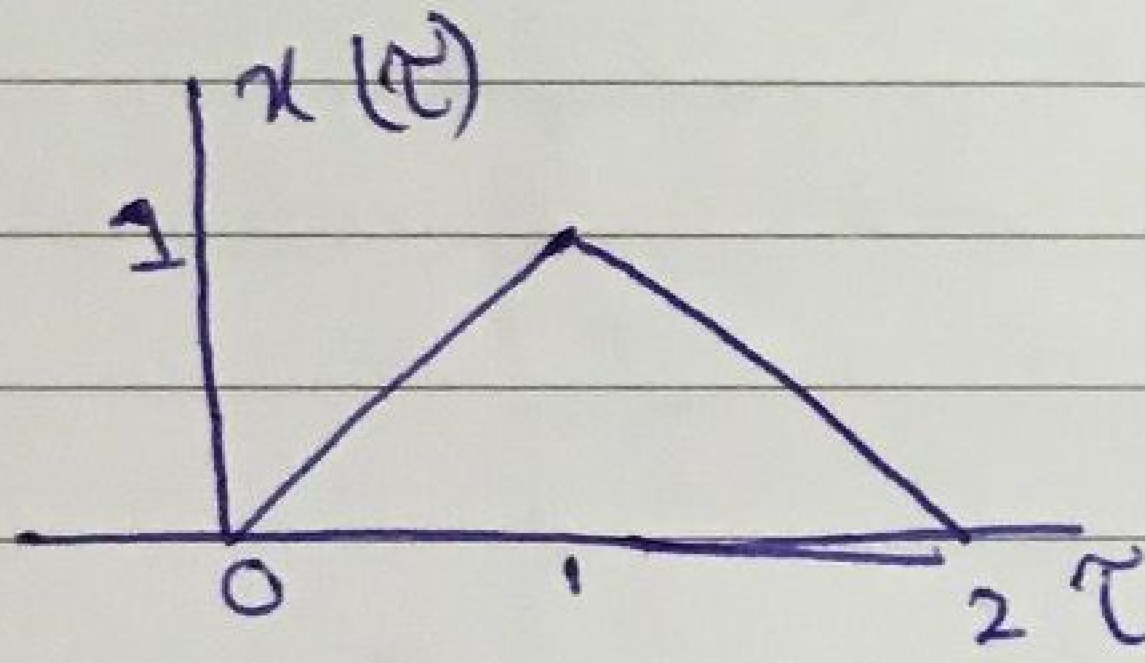
Convolve the following continuous signals:

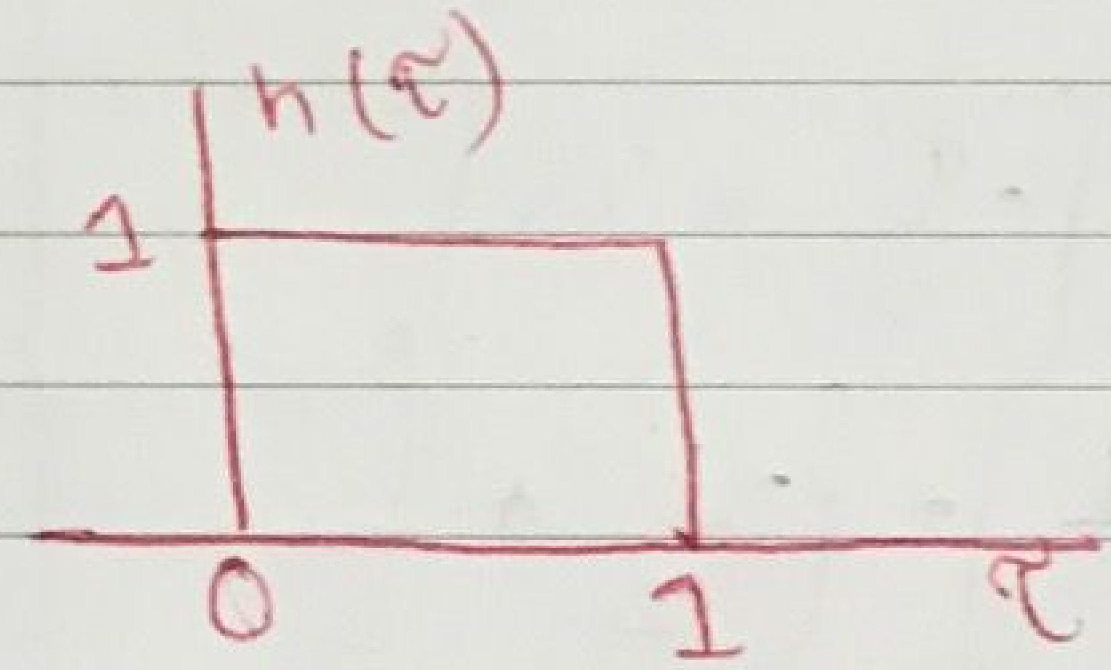




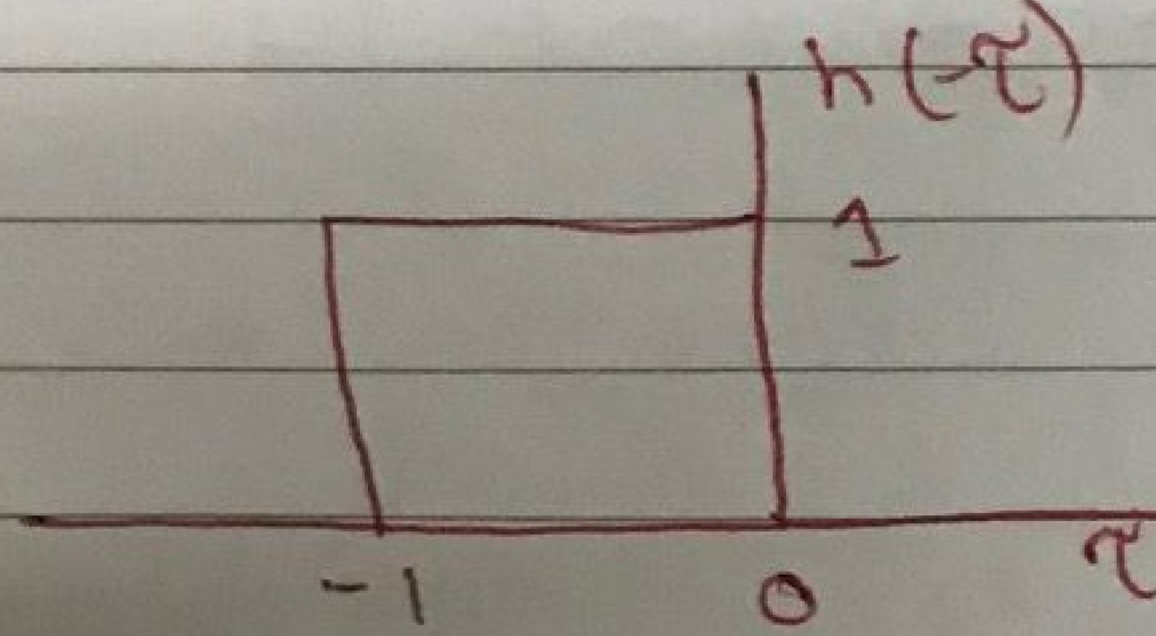
Sol:

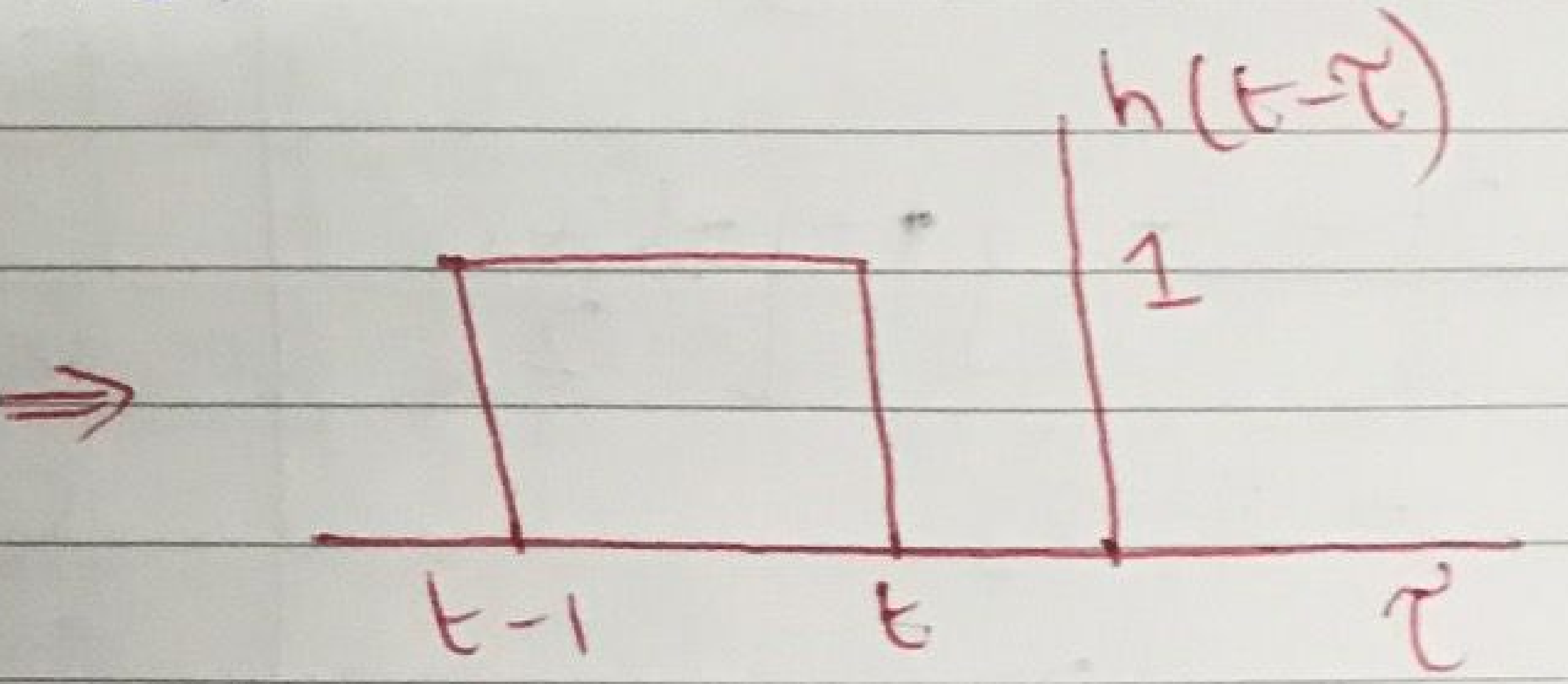
Step 1: Change $t \to \tau$





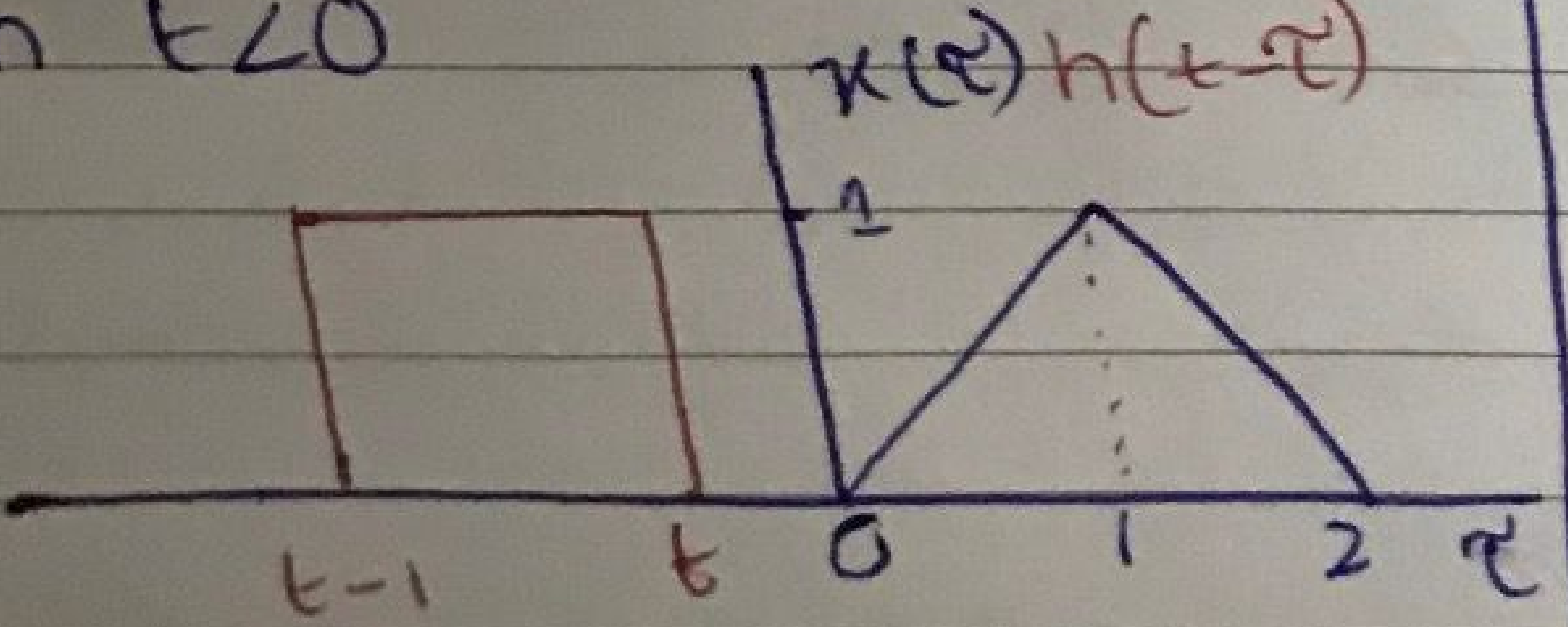
Step 2: Flip and shift $h(\tau)$.





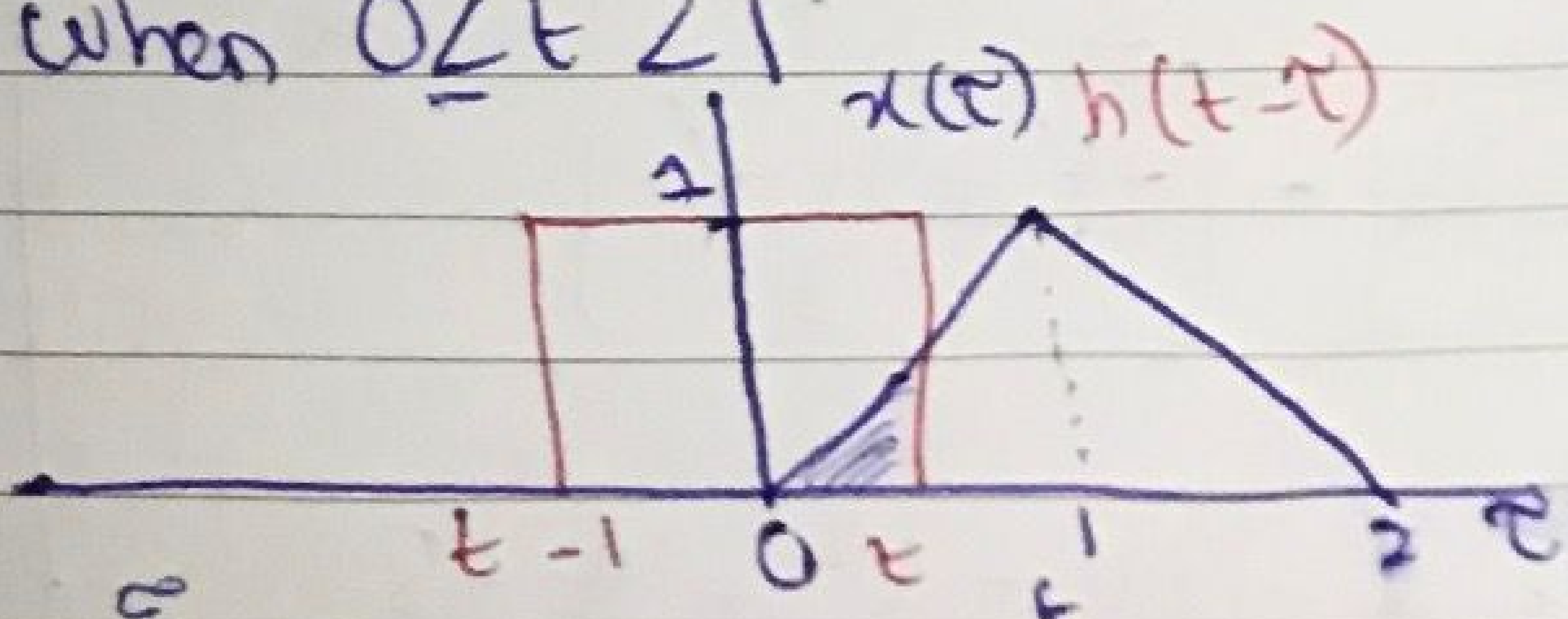
Step 3: Now convolve the two signals.

when $t < 0$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = 0,$$ ~~no overlapp~~ no overlapping

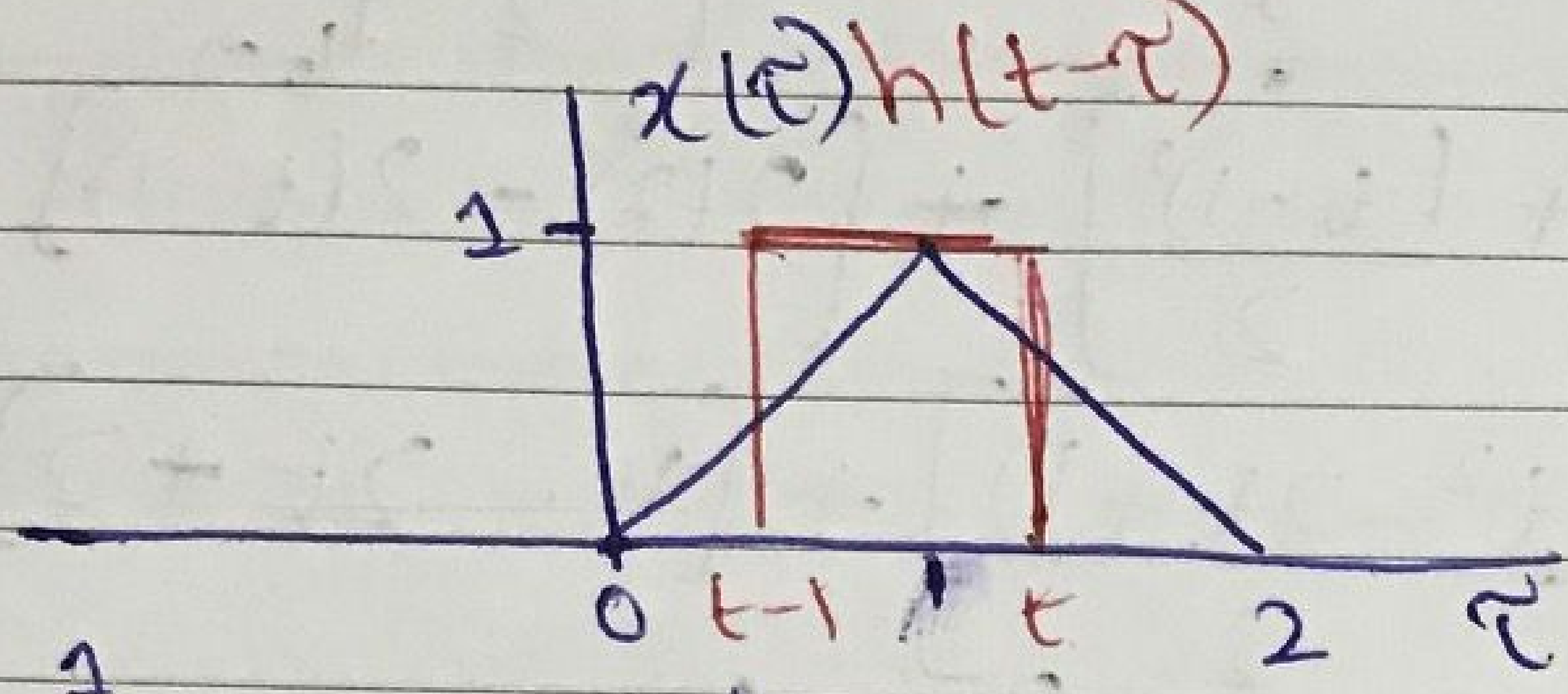
when $0 \le t < 1$



$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{0}^{t} \tau\,d\tau$$

$$= \left|\frac{\tau^2}{2}\right|_0^t = \left[\frac{t^2}{2} - 0\right] \Rightarrow \frac{t^2}{2}$$

$\Rightarrow$ when $1 \le t < 2$



$$y(t) = \int_{t-1}^{1} \tau\, d\tau + \int_{1}^{t} (-\tau+2)\, d\tau$$

$$= \int_{t-1}^{1} \tau\, d\tau + \int_{1}^{t} (-\tau)\, d\tau + 2\int_{1}^{t} d\tau$$

$$= \left|\frac{\tau^2}{2}\right|_{t-1}^{1} + \left[\frac{-\tau^2}{2}\right]_1^t + \left|2\tau\right|_1^t$$

$$= \left[\frac{1}{2} - \frac{(t-1)^2}{2}\right] + \left[-\frac{t^2}{2} + \frac{1}{2}\right] + \left[2t - 2\right]$$

$$= \left[\frac{1}{2} - \frac{(t^2-2t+1)}{2}\right] + \left[\frac{-t^2+1}{2}\right] + \left[2t - 2\right]$$

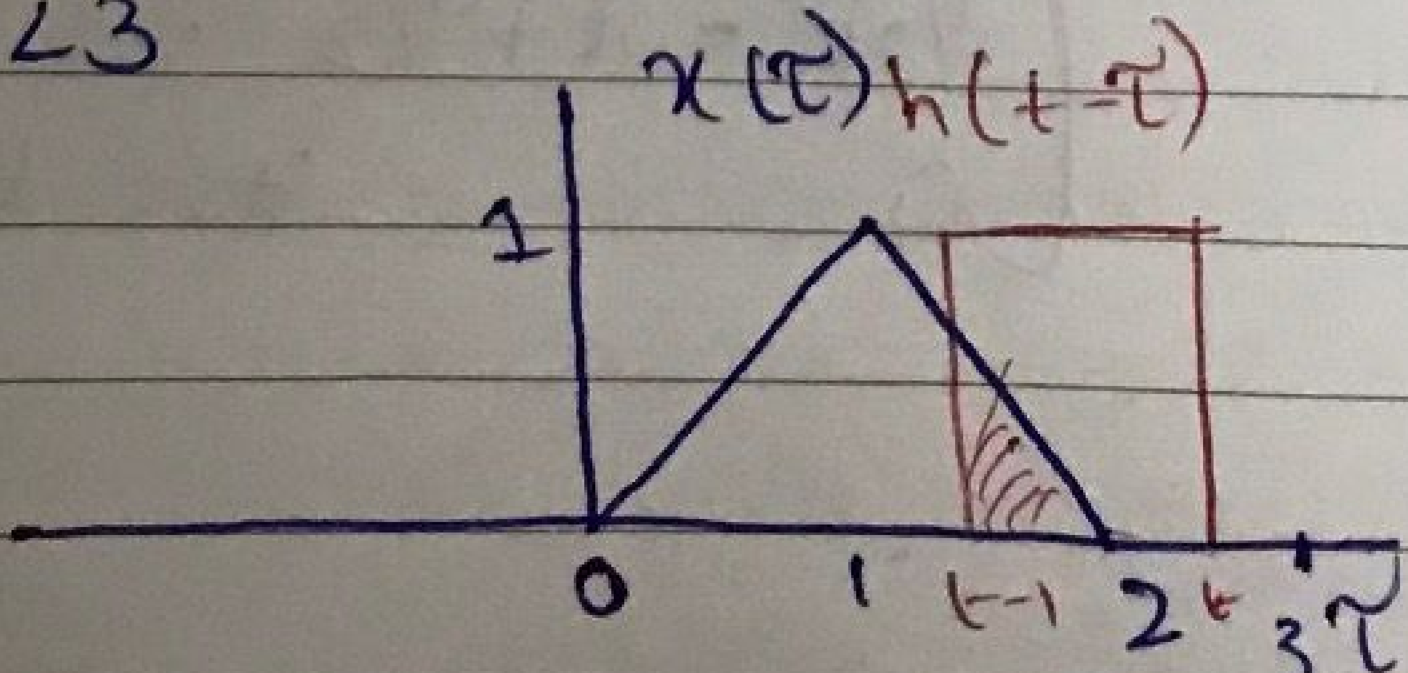
$$= \left[\frac{\cancel{1} - t^2 + 2t - \cancel{1}}{2}\right] + \left[\frac{-t^2+1+4t-4}{2}\right]$$

$$= \left[\frac{-t^2+2t}{2}\right] + \left[\frac{-t^2+4t-3}{2}\right] = \frac{(-t^2+2t)+(-t^2+4t-3)}{2}$$

$$= \frac{-t^2+2t-t^2+4t-3}{2} = \frac{-2t^2+6t-3}{2}$$

$$y(t) = -\frac{\cancel{2}t^2}{\cancel{2}_1} + \frac{{}^3\cancel{6}t}{\cancel{2}_1} - \frac{3}{2} \Rightarrow -t^2 + 3t - \frac{3}{2}$$

$\Rightarrow$ when $2 \le t < 3$



$$y(t) = \int_{t-1}^{2} (-\tau+2)\, d\tau$$

$$= -\int_{t-1}^{2} \tau \, d\tau + 2\int_{t-1}^{2} d\tau = \left| -\frac{\tau^2}{2} \right|_{t-1}^{2} + \left| 2\tau \right|_{t-1}^{2}$$

$$= \left[\frac{-(2^2)}{2} + \frac{(t-1)^2}{2} \right] + \left[2(2) - 2(t-1) \right]$$

$$= \left[-\frac{4}{2} + \frac{(t^2-2t+1)}{2} \right] + \left[4 - 2t + 2 \right]$$

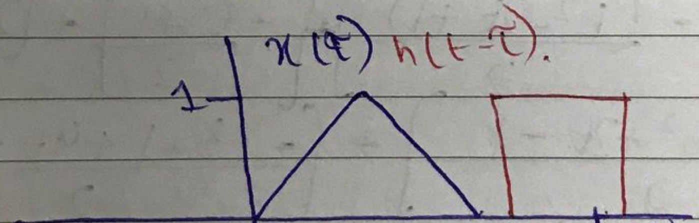
$$= \left[\frac{-4 + t^2 - 2t + 1}{2} \right] + \left(6 - 2t \right)$$

$$= \left[\frac{t^2-2t-3}{2} \right] + (6-2t) = \frac{t^2-2t-3+12-4t}{2}$$

$$y(t) = \frac{t^2 - 6t + 9}{2} = \frac{t^2}{2} - \frac{\overset{3}{\cancel{6}}t}{\cancel{2}_1} + \frac{9}{2}$$

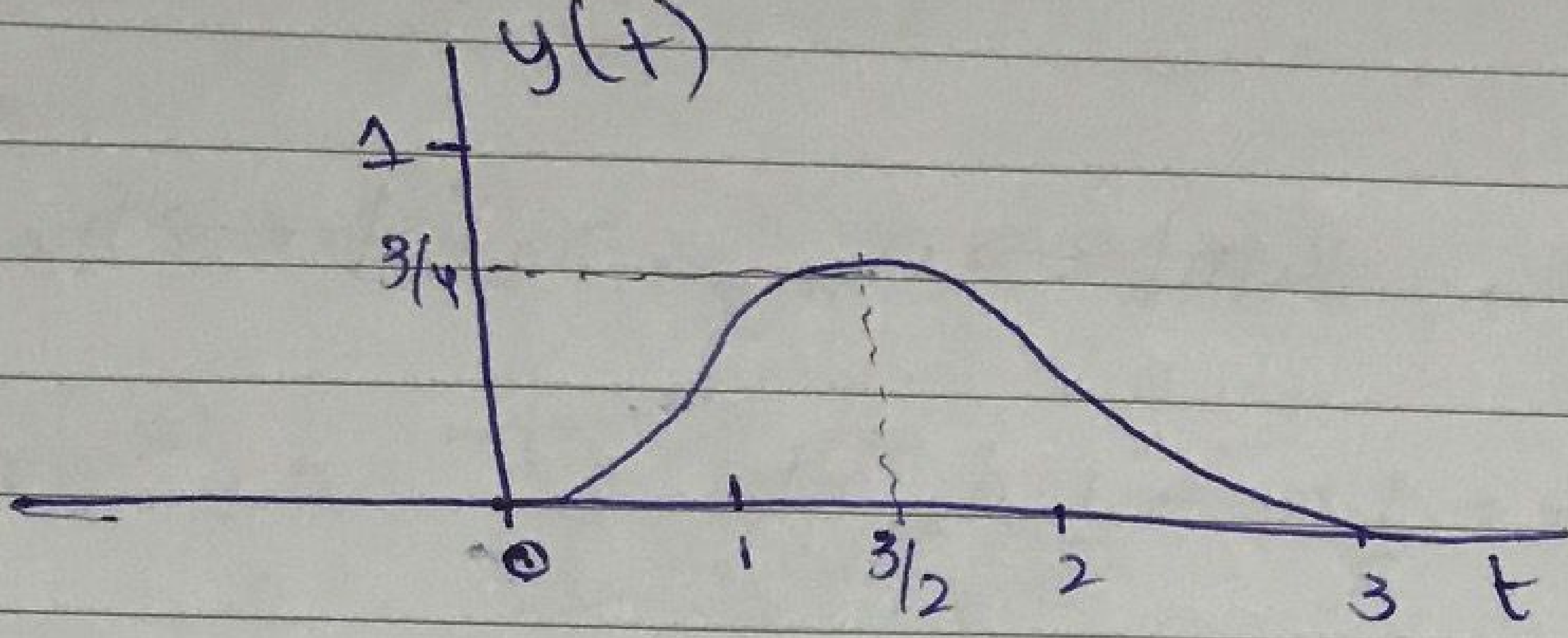
$$y(t) = \frac{t^2}{2} - 3t + \frac{9}{2}$$

$\Rightarrow$ when $t \geq 3$



$$\int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = 0 \text{ , as no overlapping}$$

$$y(t) = x(t) * h(t) \Rightarrow \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2}t^2 & \text{for } 0 \leq t < 1 \\ -t^2 + 3t - 3/2 & \text{for } 1 \leq t < 2 \\ \frac{1}{2}t^2 - 3t + 9/2 & \text{for } 2 \leq t < 3 \\ 0 & \text{for } t > 0 \end{cases}$$



PROBLEM #2:-

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\omega_0 = ?$, $a_k = ?$

Sol:-

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} , \quad \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} \Rightarrow -2je^{j\theta}+2je^{-j\theta}$$

$$x(t) = 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + \underbrace{2je^{-j(5\pi/3)t}}_{a_{-5}^{*} \text{ as it is the conjugate of } a_5}$$

$\omega_0 = \frac{\pi}{3}$

The non zero Fourier series coefficients of $x(t)$ are:

$$a_0 = 2 , \quad a_2 = a_{-2} = \frac{1}{2} , \quad a_5 = a_{-5}^{*} = -2j$$

PROBLEM #3:-

$N=4, \quad x_1[n] \leftrightarrow a_k \quad , x_2[n] \leftrightarrow b_k$

$a_0 = a_3 = \frac{1}{2} a_1 = \frac{1}{2} a_2 = 1$

$b_0 = b_1 = b_2 = b_3 = 1$

$c_k = ? \qquad g[n] = x_1[n]\, x_2[n]$

Sol:-

Using the multiplication property: $x[n]y[n] \leftrightarrow d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$

$x_1[n]\, x_2[n] \overset{F.S}{\longleftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l} = \sum_{k=0}^{3} a_l b_{k-l}$

$\overset{F.S}{\longleftrightarrow} a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3}$

$\overset{F.S}{\longleftrightarrow} b_k + 2b_{k-1} + 2b_{k-2} + b_{k-3}$

$\because a_0 = 1, a_3 = 1$

$a_1 = 2, a_2 = 2$

Since b_k is 1 for all values of k, it is clear that $b_k + 2b_{k-1} + 2b_{k-2} + b_{k-3}$ will be 6 for all values of k.

Therefore, $\quad x_1[n]\, x_2[n] \overset{FS}{\longleftrightarrow} 6$, for all k.

PROBLEM #4:

$$h_1[n] = \left(\frac{1}{3}\right)^n \quad , \quad h_2[n] = ?$$

$$H(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}}$$

Sol:-

When two LTI systems are connected in parallel, the impulse response of the overall system is the sum of the impulse responses of the individual systems. Therefore,

$$h[n] = h_1[n] + h_2[n]$$

Using the linearity property:

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Given that $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$, we obtain

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

Therefore,

$$H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega})$$

$$H_2(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}} - \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$= \underbrace{\frac{-12+5e^{-j\omega}}{(e^{-j\omega}-3)(e^{-j\omega}-4)}}_{a} - \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$a \Rightarrow \frac{-12+5e^{-j\omega}}{(e^{-j\omega}-3)(e^{-j\omega}-4)} = \frac{A}{(e^{-j\omega}-3)} + \frac{B}{(e^{-j\omega}-4)} \rightarrow (1)$$

Cross multiplication gives:

$$-12+5e^{-j\omega} = A(e^{-j\omega}-4)+B(e^{-j\omega}-3)$$

let $e^{-j\omega}=4$, $e^{-j\omega}=3$

When $e^{-j\omega}=4$

$$-12+5(4) = A(4-4)+B(4-3)$$
$$-12+20 = A(0)+B(1)$$
$$B \Rightarrow 8$$

when $e^{-j\omega}=3$

$$-12+5(3) = A(3-4)+B(3-3)$$
$$-12+15 = A(-1)+B(0)$$
$$A=-3$$

Now put A and B in equ(1)

$$a \Rightarrow \frac{-3}{e^{-j\omega}-3}+\frac{8}{e^{-j\omega}-4}$$

$$H_2(e^{j\omega}) = \frac{-3}{e^{-j\omega}-3}+\frac{8}{e^{-j\omega}-4}-\frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$= \frac{-3}{e^{-j\omega}-3}-\frac{1}{\frac{3-e^{-j\omega}}{3}}+\frac{8}{e^{-j\omega}-4}$$

$$= \frac{-3}{e^{-j\omega}-3}-\frac{3}{3-e^{-j\omega}}+\frac{8}{e^{-j\omega}-4}$$

$$= \frac{\cancel{3}}{\cancel{3-e^{-j\omega}}}-\frac{\cancel{3}}{\cancel{3-e^{-j\omega}}}+\frac{8}{e^{-j\omega}-4} \Rightarrow \frac{8}{e^{-j\omega}-4}$$

Taking 4 common from numerator and denominator

$$= \frac{\cancel{4}}{\cancel{4}}\left[\frac{2}{\frac{e^{-jw}}{4}-1}\right]$$ ∴ multiply and divide by '−'

$$H_2(e^{jw}) = \frac{-2}{1-\frac{e^{-jw}}{4}}$$

Taking inverse Fourier Transform.

$$h_2[n] = -2\left\{\left[\frac{1}{4}\right]^n u[n]\right\}$$

PROBLEM #5:

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

a) $H(e^{jw}) = ?$
b) $h[n] = ?$

Sol:-

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

a) Taking the Fourier transform of the both sides of the difference equation, we have.

$$Y(e^{jw}) - \frac{1}{6}e^{-jw}Y(e^{jw}) - \frac{1}{6}e^{-2jw}Y(e^{jw}) = X(e^{jw})$$

$$Y(e^{jw})\left[1 - \frac{1}{6}e^{-jw} - \frac{1}{6}e^{-2jw}\right] = X(e^{jw})$$

Therefore, $$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1-\frac{1}{6}e^{-jw}-\frac{1}{6}e^{-2jw}} \rightarrow (1)$$

b). Apply Partial fraction expansion on equ ①

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}} = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)}$$

$$\frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{3}e^{-j\omega}} \rightarrow ②$$

Now, cross multiplication of $\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)$ gives

$$1 = A\left(1+\frac{1}{3}e^{-j\omega}\right) + B\left(1-\frac{1}{2}e^{-j\omega}\right) \rightarrow ③$$

$$\Rightarrow 1+\frac{1}{3}e^{-j\omega} = 0 \qquad 1-\frac{1}{2}e^{-j\omega} = 0$$

$$\frac{1}{3}e^{-j\omega} = -1 \qquad -\frac{1}{2}e^{-j\omega} = -1$$

$$e^{-j\omega} = -3 \qquad e^{-j\omega} = 2$$

put $e^{-j\omega} = -3$ in equ ③

$$1 = A\left(1+\frac{1}{3}(-3)\right) + B\left[1-\frac{1}{2}(-3)\right]$$

$$1 = B\left[1+\frac{3}{2}\right]$$

$$1 = B\left[\frac{2+3}{2}\right] \Rightarrow B\left[\frac{5}{2}\right] = 1$$

$$B = \frac{2}{5}$$

Now put $e^{-j\omega} = 2$ in equ ③

$$1 = A\left[1+\frac{1}{3}(2)\right] + B\left[1-\frac{1}{2}(2)\right]$$

$$1 = A\left[\frac{3+2}{3}\right]$$

$$1 = A\left[\frac{5}{3}\right] \Rightarrow A = \frac{3}{5}$$

Now put value of A and B in equ (2)

$$H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{3}e^{-j\omega}\right)} = \frac{3/5}{1-\frac{1}{2}e^{-j\omega}} + \frac{2/5}{1+\frac{1}{3}e^{-j\omega}}$$

$$\therefore \; a^n u[n] = \frac{1}{1-\frac{1}{a}e^{-j\omega}}$$

Taking inverse transform

$$h[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n]$$

PROBLEM #6

$$x[n] = \left(\frac{1}{5}\right)^n u[n-3]$$

Z-transform = ? ROC = ?

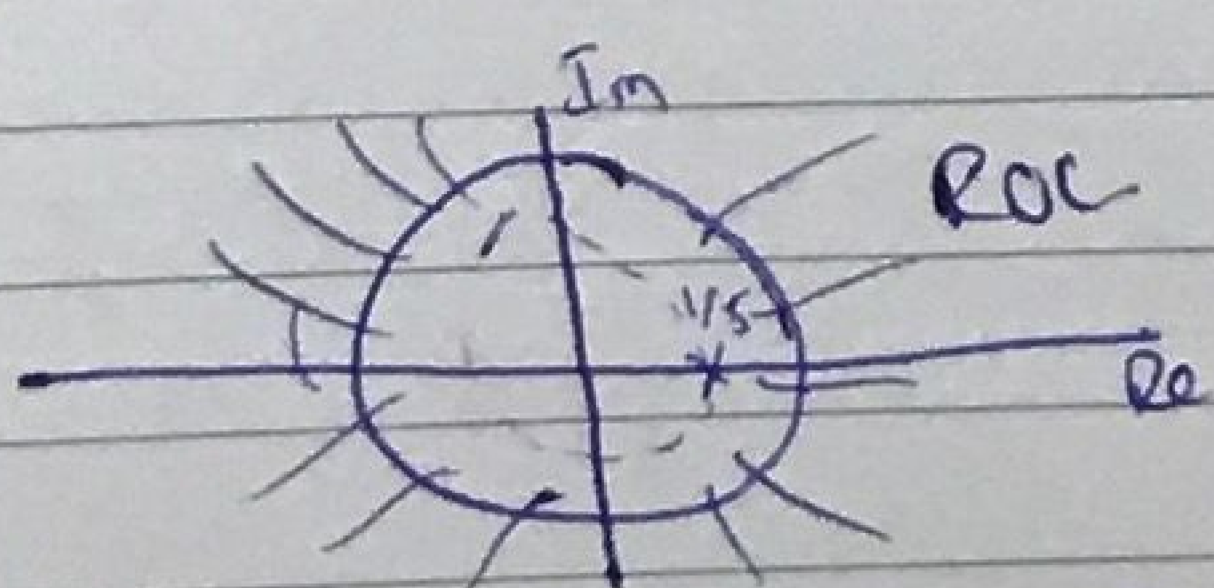
Sol

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u[n-3] z^{-n}$$

$$= \sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^n z^{-n}$$

$$= \left[\frac{z^{-3}}{125}\right] \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n}$$

$$= \left(\frac{z^{-3}}{125}\right) \frac{1}{1-\frac{1}{5}z^{-1}}, \quad |z| > \frac{1}{5}$$



PROBLEM #7:

$$X(z) = \frac{1+z^{-1}}{1+\frac{1}{3}z^{-1}}$$

a) ROC $\Rightarrow |z| > \frac{1}{3}$, $x[0], x[1], x[2] = ?$

Sol:

Since $|z| > \frac{1}{3}$, we may use long division to obtain the power series expansion of X.

$$\begin{array}{r|l} & 1 + \frac{2}{3}z^{-1} + \frac{2}{9}z^{-2} + \cdots \\ 1+\frac{1}{3}z^{-1} & 1+z^{-1} \\ & -1 - \frac{1}{3}z^{-1} \\ \hline & \frac{2}{3}z^{-1} \\ & -\frac{2}{3}z^{-1} - \frac{2}{9}z^{-2} \\ \hline & -\frac{2}{9}z^{-2} \\ & +\frac{2}{9}z^{-2} + \frac{2}{27}z^{-3} \\ \hline \end{array}$$

$$X(z) = 1 + \frac{2}{3}z^{-1} + \frac{2}{9}z^{-2} + \cdots$$

$$x[0] = 1, \quad x[1] = \frac{2}{3}, \quad x[2] = \frac{2}{9}$$

b) ROC $\Rightarrow |z| < \frac{1}{3}$, $x[0], x[-1], x[-2] = ?$

Again long division

$$\begin{array}{r|l} & 3-6z^{+1}+18z^2 \\ \hline \frac{1}{3}z^{-1}+1 & z^{-1}+1 \\ & -z^{-1}-3 \\ \hline & -2 \\ & +2+6z^{+1} \\ \hline & 6z \\ & -6z-18z^2 \\ \hline \end{array}$$

Comparing $X(z)$ with the definition of the z-transform,

$$x[0]=3, \quad x[-1]=-6, \quad x[-2]=18.$$

PROBLEM # 8:

$$X(z)=\frac{1-2z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}$$

$x[n]=?$

Sol:-

Using partial fractions.

$$X(z)=\frac{1-2z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}=\frac{A}{1-\frac{1}{2}z^{-1}}+\frac{B}{1-2z^{-1}} \rightarrow (1)$$

Cross multiplication gives

$$\frac{(1-2z^{-1})}{\cancel{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}}\cancel{(1-\tfrac{1}{2}z^{-1})(1-2z^{-1})}=A(1-2z^{-1})+B\left(1-\frac{1}{2}z^{-1}\right)$$

$$\Rightarrow 1-2z^{-1}=0 \qquad , \qquad 1-\frac{1}{2}z^{-1}=0$$

$$z^{-1}=1/2 \qquad\qquad z^{-1}=2$$

$$1-2z^{-1} = A(1-2z^{-1}) + B\left(1-\frac{1}{2}z^{-1}\right) \rightarrow ②$$

let $z^{-1} = 2$ in equ ②

$$1-2(2) = A(1-2(2)) + B\cancelto{0}{\left(1-\frac{1}{2}(2)\right)}$$

$$1-4 = A(1-4)$$

$$A(-3) = -3$$

$$A = 1$$

let $z^{-1} = \frac{1}{2}$ in equ ②

$$1-\cancel{2}\left(\frac{1}{\cancel{2}}\right) = A\cancelto{0}{\left(1-2\left(\frac{1}{2}\right)\right)} + B\left(1-\frac{1}{2}\left(\frac{1}{2}\right)\right)$$

$$1-1 = B\left[1-\frac{1}{4}\right]$$

$$0 = B\left[\frac{4-1}{4}\right] \Rightarrow B\left[\frac{3}{4}\right] = 0$$

$$B \Rightarrow 0$$

Now put values of A and B in equ ①

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{0}{1-2z^{-1}}$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

If $x[n]$ is absolutely summable, then the ROC of $X(z)$ has to include the unit circle.

Therefore, the ROC is $|z| > 1/2$. It follows that,

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$